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## TOPICS

- Vectors and Matrices
- Linear Equations
- Vector Spaces
- Orthogonality
- Determinants
- Eigenvalues and Eigenvectors
- Complex Vectors and Matrices

# I. VECTORS AND MATRICES

## I.1. *Vectors and Linear Combinations*

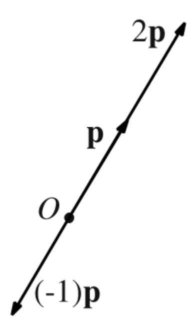
$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T \quad \text{(I.1)}$$

## I.2. *Operations on Vectors*

Let

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \underline{v} + \underline{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} = \underline{w} + \underline{v} \quad \text{(I.2)}$$

$$c\underline{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \quad \text{(I.3)}$$



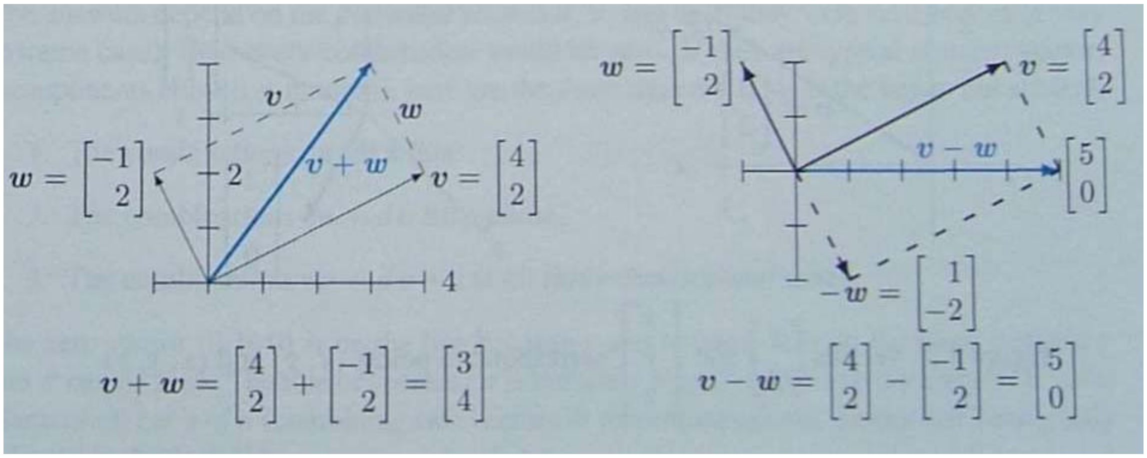

***Zero vector***

$$\underline{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{(I.4)}$$

***Linear Combinations***

If $c$ and $d$ are constants, then a linear combination of $\underline{v}$ and $\underline{w}$ is given by:

$$\underline{x} = c\underline{v} + d\underline{w} \quad \text{(I.5)}$$




When $\underline{v}$ and $\underline{w}$ are not along the same line, their linear combinations span the whole two dimensional plane.

**Linear combination**
**Multiply by 1, 4, −2**
**Then add**

$$\begin{bmatrix}1\\0\\3\end{bmatrix}+4\begin{bmatrix}1\\2\\1\end{bmatrix}-2\begin{bmatrix}2\\3\\-1\end{bmatrix}=\begin{bmatrix}1\\2\\9\end{bmatrix}$$

For one vector $\underline{u}$, the only linear combinations are the multiples $c\underline{u}$. For two vectors, the combinations are $c\underline{u}+d\underline{v}$. For three vectors, the combinations are $c\underline{u}+d\underline{v}+e\underline{w}$.

1. The combinations $c\boldsymbol{u}$ fill a ***line***.
2. The combinations $c\boldsymbol{u}+d\boldsymbol{v}$ fill a ***plane***.
3. The combinations $c\boldsymbol{u}+d\boldsymbol{v}+e\boldsymbol{w}$ fill ***three-dimensional space***.

***Spaces***

$$\mathbb{R}^i=\begin{cases}\mathbb{R}^1, & \text{Line}\\ \mathbb{R}^2, & \text{Plane}\\ \mathbb{R}^3, & \text{3D Space}\end{cases} \qquad \text{(I.6)}$$

***Example***

Let $\underline{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \underline{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow c\underline{a}_1 + d\underline{a}_2 = \begin{bmatrix} d \\ c+d \\ c \end{bmatrix} \Rightarrow$ ***Plane in*** $\mathbb{R}^3$ .

Find a vector that is not a linear combination of $\underline{a}_1, \underline{a}_2$.

Answer: $\underline{g} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}. \Rightarrow \underline{a}_1, \underline{a}_2$ do not span the whole of $\mathbb{R}^3$.

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is perpendicular to the plane. Note that for all $c$ and $d$ we have

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} . \begin{bmatrix} d \\ c+d \\ c \end{bmatrix} = d + (-1)(c+d) + c$$
$$= 0$$

**The zero vector is a linear combination and it belongs to the plane.**

**1.1 B** For $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (0, 1)$, describe all points $c\boldsymbol{v}$ with (**1**) *whole numbers* $c$ (**2**) *nonnegative* $c \geq 0$. Then add all vectors $d\,\boldsymbol{w}$ and describe all $c\boldsymbol{v} + d\,\boldsymbol{w}$.

**Solution**

(**1**) The vectors $c\boldsymbol{v} = (c, 0)$ with whole numbers $c$ are **equally spaced points** along the $x$ axis (the direction of $\boldsymbol{v}$). They include $(-2, 0)$, $(-1, 0)$, $(0, 0)$, $(1, 0)$, $(2, 0)$.

(**2**) The vectors $c\boldsymbol{v}$ with $c \geq 0$ fill a ***half-line***. It is the *positive* $x$ axis. This half-line starts at $(0, 0)$ where $c = 0$. It includes $(\pi, 0)$ but not $(-\pi, 0)$.

(**1′**) Adding all vectors $d\,\boldsymbol{w} = (0, d)$ puts a vertical line through those points $c\boldsymbol{v}$. We have infinitely many ***parallel lines*** from (*whole number* $c$, *any number* $d$).

(**2′**) Adding all vectors $d\,\boldsymbol{w}$ puts a vertical line through every $c\boldsymbol{v}$ on the half-line. Now we have a ***half-plane***. It is the right half of the $xy$ plane (any $x \geq 0$, any height $y$).

**1.1 C** Find two equations for the unknowns $c$ and $d$ so that the linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ equals the vector $\boldsymbol{b}$:

$$\boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \boldsymbol{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

**Solution** In applying mathematics, many problems have two parts:

1 *Modeling part* Express the problem by a set of equations.

2 *Computational part* Solve those equations by a fast and accurate algorithm.

Here we are only asked for the first part (the equations). Chapter 2 is devoted to the second part (the algorithm). Our example fits into a fundamental model for linear algebra:

$$\text{Find} \quad c_1, \ldots, c_n \quad \text{so that} \quad c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{b}.$$

For $n = 2$ we could find a formula for the $c$'s. The "elimination method" in Chapter 2 succeeds far beyond $n = 100$. For $n$ greater than 1 million, see Chapter 9. Here $n = 2$:

**Vector equation** $$c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The required equations for $c$ and $d$ just come from the two components separately:

**Two scalar equations** $$\begin{aligned} 2c - d &= 1 \\ -c + 2d &= 0 \end{aligned}$$

You could think of those as two lines that cross at the solution $c = \frac{2}{3}, d = \frac{1}{3}$.

## I.3. Dot Products and Lengths

### I.3.A. Dot Product

$$\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \Rightarrow \underline{a} \cdot \underline{b} = \underline{a}^T \underline{b} = \sum_{i=1}^{n} a_i b_i \tag{I.7}$$

***Example***

Let $\underline{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \underline{a}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \underline{a}_1 \cdot \underline{a}_2 = (1)(-1) + (-2)(-2) + (3)(1) = 6$.

***Example***

Let $\underline{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \underline{a}_2 = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \Rightarrow \underline{a}_1 \cdot \underline{a}_2 = (1)(-1) + (-2)(-2) + (3)(-1) = 0$.

$\underline{a}_1$ and $\underline{a}_2$ are perpendicular.

$$\underline{a}_1 \perp \underline{a}_2$$

$$\underline{b} \cdot \underline{a} = \underline{a} \cdot \underline{b} \quad \text{(I.8)}$$

## I.3.B. Length

$$\underline{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \Rightarrow \underline{a} \cdot \underline{a} = \underline{a}^T \underline{a} = \sum_{i=1}^{n} a_i^2 = \|\underline{a}\|^2 \quad \text{(I.9)}$$

***Squared Length***

$$\|\underline{a}\|^2 = \underline{a} \cdot \underline{a} = \underline{a}^T \underline{a} = \sum_{i=1}^{n} a_i^2 \quad \text{(I.10)}$$

***Length***

$$\|\underline{a}\| = \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{\underline{a}^T \underline{a}} = \sqrt{\sum_{i=1}^{n} a_i^2} \quad \text{(I.11)}$$

$\|\underline{a}\|$ is also called the norm of $\underline{a}$; $\text{norm}(\underline{a}) = \|\underline{a}\|$.

***Unit Vectors***

A unit vector $\underline{u}$ has a length of unity.

$$\|\underline{u}\| = 1$$
$$\|\underline{u}\|^2 = 1 \quad \text{(I.12)}$$

We can generate a unit vector by dividing an arbitrary vector by its own length, unless the vector length is zero.

***Example***

Let $\underline{u} = \begin{bmatrix} -2 \\ 5 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \|\underline{u}\|^2 = 39 \Rightarrow \|\underline{u}\| = \sqrt{39}$

The vector $\underline{a} = \frac{\underline{u}}{\|\underline{u}\|} = \frac{1}{\sqrt{39}} \begin{bmatrix} -2 \\ 5 \\ 1 \\ 3 \end{bmatrix}$

is a unit vector. We can use this method to generate unit vectors as long as $\|\underline{u}\| > 0$

**Example 4** The standard unit vectors along the $x$ and $y$ axes are written $\boldsymbol{i}$ and $\boldsymbol{j}$. In the $xy$ plane, the unit vector that makes an angle "theta" with the $x$ axis is $(\cos\theta, \sin\theta)$:

**Unit vectors** $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\boldsymbol{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

***Angle Between Two Vectors***

$$\underline{v}.\underline{w} = \|\underline{v}\| \, \|\underline{w}\| \cos\theta \qquad \text{(I.13)}$$

## I.3.C. Perpendicular Vectors

Vectors $\underline{a}$ and $\underline{b}$ are perpendicular when

$$\underline{a} \cdot \underline{b} = 0 \qquad \text{(I.14)}$$

This written as: $\underline{a} \perp \underline{b}$ .

When $\underline{a} \perp \underline{b} \Rightarrow$ they form the two perpendicular sides of a right triangle. $\underline{a} - \underline{b}$ forms the hypotenuse. From the Pythagoras Law,

$$\|\underline{a}\|^2 + \|\underline{b}\|^2 = \|\underline{a} - \underline{b}\|^2 \Rightarrow \underline{a} \cdot \underline{b} = 0 \qquad \text{(I.15)}$$

***Schwartz Inequality***

$$\underline{a} \cdot \underline{b} = \|\underline{a}\|\|\underline{b}\| \cos\theta_{ab} \Rightarrow |\underline{a} \cdot \underline{b}| \le \|\underline{a}\|\|\underline{b}\| \qquad \text{(I.16)}$$

***Triangle Inequalty***

$$\|\underline{a}+\underline{b}\| \le \|\underline{a}\| + \|\underline{b}\| \tag{I.17}$$

## I.4. Matrices

A matrix is a set of vectors grouped together in one mathematical entity.

A matrix consists of a number ( $n$ ) of columns:

$$A = \begin{bmatrix} \underline{a}_1 & \cdots & \underline{a}_n \end{bmatrix} \tag{I.18}$$

A matrix can also be considered to consist of a number ( $m$ ) of rows:

$$A = \begin{bmatrix} \underline{q}_1^T \\ \vdots \\ \underline{q}_m^T \end{bmatrix} \tag{I.19}$$

**First example** $\quad u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

Their linear combinations in three-dimensional space are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$:

**Combinations** $$c\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + e\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \tag{1}$$

Now something important: *Rewrite that combination using a matrix.* The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ go into the columns of the matrix $A$. That matrix "multiplies" a vector:

**Same combination is now $A$ times $x$** $$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = \begin{bmatrix} c \\ d-c \\ e-d \end{bmatrix}. \tag{2}$$

The numbers $c, d, e$ are the components of a vector $\boldsymbol{x}$. The matrix $A$ times the vector $\boldsymbol{x}$ is the same as the combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ of the three columns:

**Matrix times vector** $$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{u} & \boldsymbol{v} & \boldsymbol{w} \end{bmatrix}\begin{bmatrix} c \\ d \\ e \end{bmatrix} = c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}. \tag{3}$$

Note that $A$ is lower triangular.

This can be put as a description of a linear system with an input vector $\underline{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and an output vercor $\underline{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$, as follows

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \boldsymbol{b}. \qquad (4)$$

The input is $\boldsymbol{x}$ and the output is $\boldsymbol{b} = A\boldsymbol{x}$. This $A$ is a "**difference matrix**" because $\boldsymbol{b}$ contains differences of the input vector $\boldsymbol{x}$. The top difference is $x_1 - x_0 = x_1 - 0$.

## I.4.A. Linear Equations

The input-output relationship of a linear system with an input $\underline{x}$, output $\underline{b}$ and coefficient matrix $A$ can be written in the form

$$A\underline{x} = \underline{b} \qquad \text{(I.20)}$$

Usually, it is required to find an input vector $\underline{x}$, given an output vector $\underline{b}$ has been observed.

| | | | |
|---|---|---|---|
| $A\boldsymbol{x} = \boldsymbol{b}$ | $x_1 = b_1$<br>$-x_1 + x_2 = b_2$<br>$-x_2 + x_3 = b_3$ | **Solution** | $x_1 = b_1$<br>$x_2 = b_1 + b_2$<br>$x_3 = b_1 + b_2 + b_3$. |

Having found a solution to $A\underline{x} = \underline{b}$ means that $A$ is invertible.

When $\underline{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

When $\underline{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$.

We can have $C\underline{x} = 0$ even if $C \neq 0, \underline{x} \neq 0$.

## I.4.B. The Inverse Matrix

The solution to $A\underline{x} = \underline{b}$ is

$$\underline{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_1+b_2\\b_1+b_2+b_3\end{bmatrix}=\begin{bmatrix}1&&\\1&1&\\1&1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=A^{-1}\underline{b} \qquad \text{(I.21)}$$

## I.4.C. Independence and Dependence

Two vectors $\underline{a}$ and $\underline{b}$ that are not on the same line ($\underline{a}\neq c\underline{b}$) span a plane.

Consider a third vector $\underline{d}$.

If $\underline{d}$ is in the plane, then it is dependent on (linear combination of) $\underline{a}$ and $\underline{b}$.

If $\underline{d}$ is not in the plane, then it is independent of $\underline{a}$ and $\underline{b}$.

We cannot have three independent vectors in $\mathbb{R}^2$.

For independent vectors $\underline{\gamma}_1,\underline{\gamma}_2,\underline{\gamma}_3$, $c_1\underline{\gamma}_1+\cdots+c_n\underline{\gamma}_n=0$ is true only when $c_1=\cdots=c_n=0$.

**1.3 B** This $E$ is an **elimination matrix**. $E$ has a subtraction, $E^{-1}$ has an addition.

$$E\boldsymbol{x}=\boldsymbol{b} \quad \begin{bmatrix}\mathbf{1}&0\\-\ell&\mathbf{1}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix} \qquad E=\begin{bmatrix}\mathbf{1}&0\\-\ell&\mathbf{1}\end{bmatrix}$$

The first equation is $x_1=b_1$. The second equation is $x_2-\ell x_1=b_2$. The inverse will *add* $\ell x_1=\ell b_1$, because the elimination matrix *subtracted* $\ell x_1$:

$$\boldsymbol{x}=E^{-1}\boldsymbol{b} \quad \begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\\ell b_1+b_2\end{bmatrix}=\begin{bmatrix}\mathbf{1}&0\\\ell&\mathbf{1}\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix} \qquad E^{-1}=\begin{bmatrix}\mathbf{1}&0\\\ell&\mathbf{1}\end{bmatrix}$$

# II. SOLVING LINEAR EQUATIONS

## II.1. Vectors and Linear Equations

### II.1.A. ROW PICTURE

We begin a row at a time. The first equation produces a straight line in the $xy$ plane. The point $(1,0)$ is on the line because it solves that equation. The point $(3,1)$ is also on the line. If we choose $x=101$ we find $y=50$.

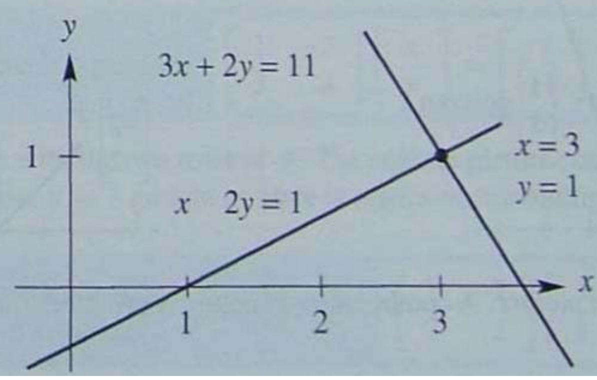
The slope of this particular line is $\frac{1}{2}$ because $y$ increases by 1 when $x$ changes by 2. But slopes are important in calculus and this is linear algebra!

The second line in this "row picture" comes from the second equation. Observe the intersection point where the two lines meet. The point $(3,1)$ lies on both lines. That point solves both equations, at once. This is the solution to our system of linear equations.

**Two equations** $\quad x - 2y = 1$
**Two unknowns** $\quad 3x + 2y = 11$ $\quad$ (1)

We begin *a row at a time*. The first equation $\mathbf{x - 2y = 1}$ produces a straight line in the $xy$ plane. The point $x = 1, y = 0$ is on the line because it solves that equation. The point $x = 3, y = 1$ is also on the line because $3 - 2 = 1$. If we choose $x = 101$ we find $y = 50$.

The slope of this particular line is $\frac{1}{2}$, because $y$ increases by 1 when $x$ changes by 2. But slopes are important in calculus and this is linear algebra!




**Row picture: two lines meeting at a single point (the solution).**

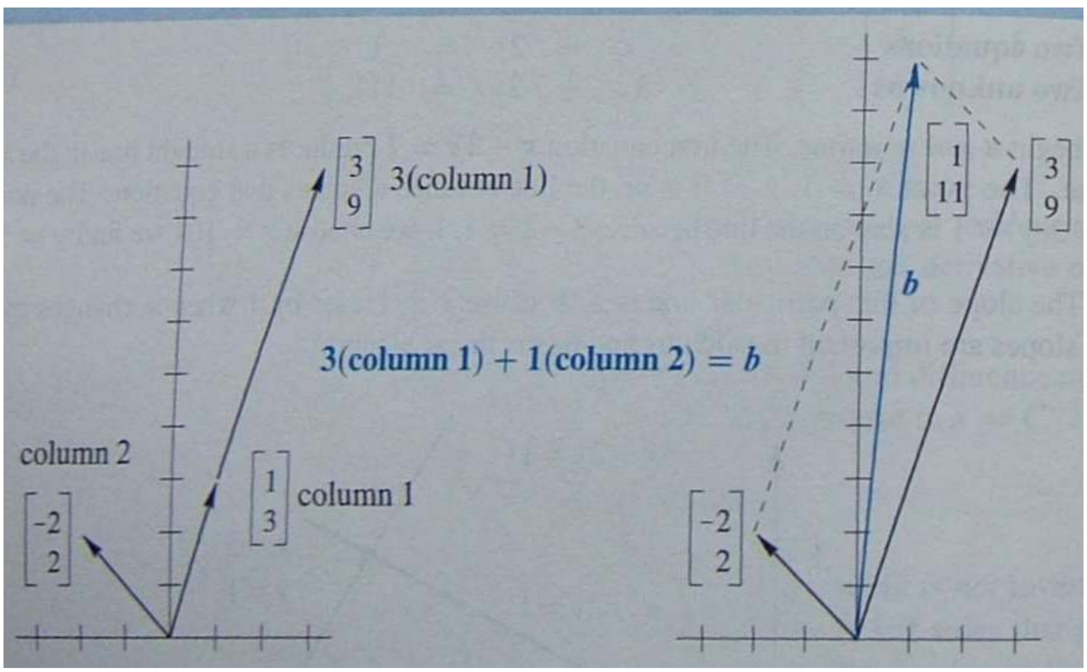
## II.1.B. COLUMN PICTURE

We recognize the same linear system as a "vector equation". Instead of numbers we need to see vectors. If you separate the original system into its columns instead of its rows, you get a vector equation

$$x\begin{bmatrix}1\\3\end{bmatrix}+y\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1\\11\end{bmatrix} \\ =\underline{b} \tag{I.22}$$

This has two column vectors on the left side. The problem is to find the combination of those vectors that equals the vector on the right.

The column picture combines the column vectors on the left side to produce the vector $\underline{b}$ on the right side.




We are multiplying the first column by $x$ and the second column by $y$, and adding. With the right choices $x=3$ and $y=1$ (the same numbers as before), this produces

$$\begin{matrix} 3(\text{column 1}) & + & 1(\text{column 2}) & = & \underline{b} \\ \begin{bmatrix}3\\9\end{bmatrix} & + & \begin{bmatrix}-2\\2\end{bmatrix} & = & \begin{bmatrix}1\\11\end{bmatrix}\end{matrix} \tag{I.23}$$

Note that if the components of a vector $\underline{v}$ are $v_1$ and $v_2$ then

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
$$c\underline{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \tag{I.24}$$

Note also that vector addition means adding the vector components separately:

$$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{I.25}$$

$$\underline{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{I.26}$$

$$\underline{v} + \underline{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \tag{I.27}$$

## II.1.C. Matrix Picture

**Coefficient matrix** $A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$.

This is very typical of linear algebra, to look at a matrix by rows and by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $A\boldsymbol{x} = \boldsymbol{b}$:

**Matrix equation** $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$.

The row picture deals with the two rows of $A$. The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into $\boldsymbol{x}$. Here is matrix-vector multiplication:

**Dot products with rows**
**Combination of columns** $A\boldsymbol{x} = \boldsymbol{b}$ is $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$.

## II.1.D. Three Equations in Three Unknowns

***Row Picture***

The three unknowns are $x, y, z$. We have three linear equations:

$$Ax = b \qquad \begin{array}{rcrcrcl} x & + & 2y & + & 3z & = & 6 \\ 2x & + & 5y & + & 2z & = & 4 \\ 6x & - & 3y & + & z & = & 2 \end{array}$$

Each equation produces a plane in three-dimensional space. That plane crosses the x and y and z axes at the points (6,0,0) and (0,3,0) and (0,0,2). Those three points solve the equation and they determine the whole plane. Note that a line is determined by specifying two points on it, while a plane is determined by specifying three points on it. Note that the intersection of two planes (e.g., first two equations) is a line (call it L). Any point on the line satisfies the first two equations. The plane of the third equation insects L at one point. This point is the solution of the system.

***Column Picture***

$$x\begin{bmatrix}1\\2\\6\end{bmatrix} + y\begin{bmatrix}2\\5\\-3\end{bmatrix} + z\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}.$$

Correct combination (solution):

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix} \qquad \text{(I.28)}$$

***Matrix Picture***

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix} \qquad \text{(I.29)}$$

$$A\underline{x} = \underline{b} \qquad \text{(I.30)}$$

We can perform multiplication by rows or multiplication by columns:

***Multiplication by rows*** $Ax$ comes from *dot products*, each row times the column $x$:

$$Ax = \begin{bmatrix} (\textit{row 1}) \cdot x \\ (\textit{row 2}) \cdot x \\ (\textit{row 3}) \cdot x \end{bmatrix}.$$

***Multiplication by columns*** $Ax$ is a ***combination of column vectors***:

$$Ax = x\,(\textit{column 1}) + y\,(\textit{column 2}) + z\,(\textit{column 3}).$$

Identity Matrix

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \tag{I.31}$$

***Matrix Notation***

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{I.32}$$

**2.1 A** Describe the column picture of these three equations $Ax = b$. Solve by careful inspection of the columns (instead of elimination):

$$\begin{aligned} x + 3y + 2z &= -3 \\ 2x + 2y + 2z &= -2 \\ 3x + 5y + 6z &= -5 \end{aligned} \quad \text{which is} \quad \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 2 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -5 \end{bmatrix}.$$

**Solution** The column picture asks for a linear combination that produces $b$ from the three columns of $A$. In this example $b$ is *minus the second column*. So the solution is $x = 0, y = -1, z = 0$. To show that $(0, -1, 0)$ is the *only* solution we have to know that "$A$ is invertible" and "the columns are independent" and "the determinant isn't zero."

Those words are not yet defined but the test comes from elimination: We need (and for this matrix we find) a full set of three nonzero pivots.

Suppose the right side changes to $b = (4, 4, 8) =$ sum of the first two columns. Then the good combination has $x = 1, y = 1, z = 0$. The solution becomes $x = (1, 1, 0)$.

## System with No Solution

**2.1 B** This system has *no solution*. The planes in the row picture don't meet at a point. ***No combination of the three columns produces b. How to show this?***

$$\begin{aligned} x + 3y + 5z &= 4 \\ x + 2y - 3z &= 5 \\ 2x + 5y + 2z &= 8 \end{aligned} \qquad \begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & -3 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} = \boldsymbol{b}$$

**(1)** Multiply the equations by $1, 1, -1$ and add to get $0 = 1$. *No solution*. Are any two of the planes parallel? What are the equations of planes parallel to $x + 3y + 5z = 4$?

**(2)** Take the dot product of each column of $A$ (and also $\boldsymbol{b}$) with $\boldsymbol{y} = (1, 1, -1)$. How do those dot products show that the system $A\boldsymbol{x} = \boldsymbol{b}$ has no solution?

**(3)** Find three right side vectors $\boldsymbol{b}^*$ and $\boldsymbol{b}^{**}$ and $\boldsymbol{b}^{***}$ that *do* allow solutions.

**Solution**

**(1)** Multiplying the equations by $1, 1, -1$ and adding gives $0 = 1$:

$$\begin{aligned} x + 3y + 5z &= 4 \\ x + 2y - 3z &= 5 \\ -[2x + 5y + 2z &= 8] \\ \hline 0x + 0y + 0z &= 1 \qquad \textit{No Solution} \end{aligned}$$

The planes don't meet at a point, even though no two planes are parallel. For a plane parallel to $x + 3y + 5z = 4$, change the "4". The parallel plane $x + 3y + 5z = 0$ goes through the origin $(0, 0, 0)$. And the equation multiplied by any nonzero constant still gives the same plane, as in $2x + 6y + 10z = 8$.

**(2)** The dot product of each column of $A$ with $\boldsymbol{y} = (1, 1, -1)$ is *zero*. On the right side, $\boldsymbol{y} \cdot \boldsymbol{b} = (1, 1, -1) \cdot (4, 5, 8) = 1$ is *not zero*. So a solution is impossible.

**(3)** There is a solution when $\boldsymbol{b}$ is a combination of the columns. These three choices of $\boldsymbol{b}$ have solutions $\boldsymbol{x}^* = (1, 0, 0)$ and $\boldsymbol{x}^{**} = (1, 1, 1)$ and $\boldsymbol{x}^{***} = (0, 0, 0)$:

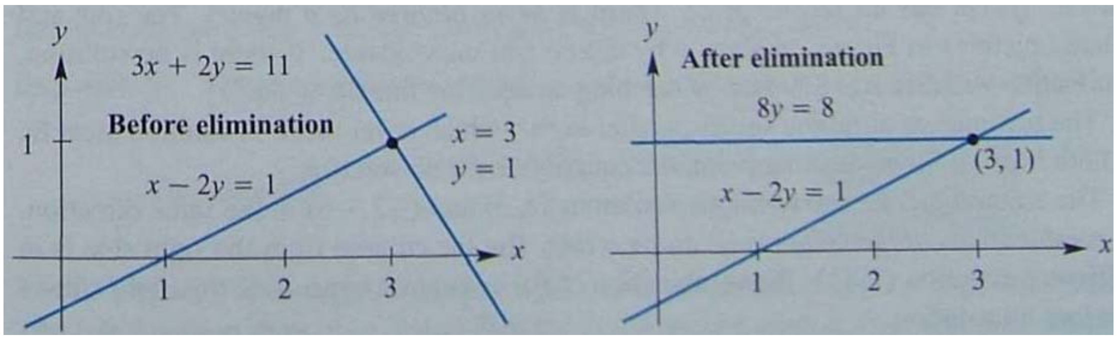
$$\boldsymbol{b}^* = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \text{first column} \qquad \boldsymbol{b}^{**} = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} = \text{sum of columns} \qquad \boldsymbol{b}^{***} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

## II.2. Elimination

### II.2.A. Procedure

| | | | | |
|---|---|---|---|---|
| **Before** | $x - 2y = 1$<br>$3x + 2y = 11$ | **After** | $x - 2y = 1$<br>$8y = 8$ | (*multiply equation 1 by 3*)<br>(*subtract to eliminate 3x*) |

Pivots: 1, 8.




***Permanent failure with no solution.*** Elimination makes this clear:

| | | |
|---|---|---|
| $x - 2y = 1$<br>$3x - 6y = 11$ | Subtract 3 times eqn. 1 from eqn. 2 | $x - 2y = 1$<br>$\mathbf{0}y = 8.$ |

***Failure with infinitely many solutions. Change $b = (1, 11)$ to $(1, 3)$.***

| | | | |
|---|---|---|---|
| $x - 2y = 1$<br>$3x - 6y = 3$ | Subtract 3 times eqn. 1 from eqn. 2 | $x - 2y = 1$<br>$\mathbf{0}y = \mathbf{0}.$ | **Still only one pivot.** |

**3** ***Temporary failure (zero in pivot). A row exchange produces two pivots:***

| | | | |
|---|---|---|---|
| **Permutation** | $0x + 2y = 4$<br>$3x - 2y = 5$ | Exchange the two equations | $3x - 2y = 5$<br>$2y = 4.$ |

Let

$$\begin{aligned} 2x+4y-2z&=2 \\ 4x+9y-3z&=8 \\ -2x-3y+7z&=10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x+4y-2z&=2 \\ y+z&=4 \\ 4z&=8 \end{aligned}$$

$$A\underline{x} = \underline{b} \Rightarrow U\underline{x} = \underline{c}$$

Solution: $\underline{x} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

**2.2 C** Use elimination to reach upper triangular matrices $U$. Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in the last equation.

$$\begin{array}{rcc} \textbf{Success} & x+y+z=7 & x+y+z=7 \\ \textbf{then} & x+y-z=5 & x+y-z=5 \\ \textbf{Failure} & x-y+z=3 & -x-y+z=3 \end{array}$$

**Solution** For the first system, subtract equation 1 from equations 2 and 3 (the multipliers are $\ell_{21}=1$ and $\ell_{31}=1$). The 2, 2 entry becomes zero, so exchange equations:

$$\textbf{Success} \quad \begin{aligned} x+y+z &= 7 \\ \mathbf{0}y-2z &= -2 \\ -2y+0z &= -4 \end{aligned} \quad \text{exchanges into} \quad \begin{aligned} x+y+z &= 7 \\ -\mathbf{2}y+0z &= -4 \\ -2z &= -2 \end{aligned}$$

Then back substitution gives $z=1$ and $y=2$ and $x=4$. The pivots are $1, -2, -2$.

For the second system, subtract equation 1 from equation 2 as before. Add equation 1 to equation 3. This leaves zero in the 2, 2 entry *and also below*:

$$\textbf{Failure} \quad \begin{aligned} x+y+z &= 7 \\ \mathbf{0}y-2z &= -2 \\ \mathbf{0}y+2z &= 10 \end{aligned}$$

There is *no pivot* in column 2 (it was $-$ column 1)
A further elimination step gives $0z=8$
The three planes don't meet

Plane 1 meets plane 2 in a line. Plane 1 meets plane 3 in a parallel line. *No solution.*

If we change the "3" in the original third equation to "$-5$" then elimination would lead to $0=0$. There are infinitely many solutions! *The three planes now meet along a whole line.*

Changing 3 to $-5$ moved the third plane to meet the other two. The second equation gives $z=1$. Then the first equation leaves $x+y=6$. **No pivot in column 2 makes $y$ free** (it can have any value). Then $x=6-y$.

## II.3. Elimination Using Matrices

$$\begin{aligned} 2x_1+4x_2-2x_3 &= 2 \\ 4x_1+9x_2-3x_3 &= 8 \\ -2x_1-3x_2+7x_3 &= 10 \end{aligned} \quad \text{is the same as} \quad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

Subtracting twice the first row from the second row is equivalent to multiplying $A$ by:

$$E_{21}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\Rightarrow E_{21}A=\begin{bmatrix}\boxed{2}&4&-2\\0&\boxed{1}&1\\-2&-3&7\end{bmatrix},\ E_{21}\underline{b}=\begin{bmatrix}2\\4\\10\end{bmatrix}$$

$$E_{21}^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}$$

Note that

$$E_{31}=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\Rightarrow E_{31}E_{21}A=\begin{bmatrix}\boxed{2}&4&-2\\0&\boxed{1}&1\\0&1&5\end{bmatrix},\ E_{31}E_{21}\underline{b}=\begin{bmatrix}2\\4\\12\end{bmatrix}$$

$$E_{31}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}$$

$$E_{32}=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\Rightarrow E_{32}E_{31}E_{21}A=\begin{bmatrix}\boxed{2}&4&-2\\0&\boxed{1}&1\\0&0&\boxed{4}\end{bmatrix},\ E_{32}E_{31}E_{21}\underline{b}=\begin{bmatrix}2\\4\\8\end{bmatrix}$$

$$E_{32}^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$$

$$E=E_{32}E_{31}E_{21}$$

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\1&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\3&-1&1\end{bmatrix}$$

$$EA=\begin{bmatrix}1&0&0\\-2&1&0\\3&-1&1\end{bmatrix}\begin{bmatrix}2&4&-2\\4&9&-3\\-2&-3&7\end{bmatrix}=\begin{bmatrix}\boxed{2}&4&-2\\0&\boxed{1}&1\\0&0&\boxed{4}\end{bmatrix}$$

$$E\underline{b} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$$

Note the inverse of $E$ is

$$\begin{aligned} E &= E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \end{aligned}$$

Note that

$$EE^{-1} = E^{-1}E = I$$

| | |
|---|---|
| **Associative law is true** | $A(BC) = (AB)C$ |
| **Commutative law is false** | Often $AB \neq BA$ |

There is another requirement on matrix multiplication. Suppose $B$ has only one column (this column is $\boldsymbol{b}$). The matrix-matrix law for $EB$ should agree with the matrix-vector law for $E\boldsymbol{b}$. Even more, we should be able to *multiply matrices $EB$ a column at a time*:

***If $B$ has several columns $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$, then the columns of $EB$ are $E\boldsymbol{b}_1, E\boldsymbol{b}_2, E\boldsymbol{b}_3$.***

| | | |
|---|---|---|
| **Matrix multiplication** | $AB = A[\boldsymbol{b}_1\ \boldsymbol{b}_2\ \boldsymbol{b}_3] = [A\boldsymbol{b}_1\ A\boldsymbol{b}_2\ A\boldsymbol{b}_3]$. | (3) |

***Permutation Matrices***

$$P_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

## ***The Augmented Matrix***

***Augmented matrix*** $\quad \begin{bmatrix} A & \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 4 & 9 & -3 & \mathbf{8} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix}.$

*Elimination acts on whole rows of this matrix.* The left side and right side are both multiplied by $E$, to subtract 2 times equation 1 from equation 2. With $\begin{bmatrix} A & \boldsymbol{b} \end{bmatrix}$ those steps happen together:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 4 & 9 & -3 & \mathbf{8} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & \mathbf{2} \\ 0 & 1 & 1 & \mathbf{4} \\ -2 & -3 & 7 & \mathbf{10} \end{bmatrix}.$$

$$\begin{aligned} AB &= A\begin{bmatrix} \underline{b}_1 & \underline{b}_2 & \cdots & \underline{b}_m \end{bmatrix} \\ &= \begin{bmatrix} A\underline{b}_1 & A\underline{b}_2 & \cdots & A\underline{b}_m \end{bmatrix} \end{aligned}$$

$$A^T B = \begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_n^T \end{bmatrix} B$$

**2.3 A** What 3 by 3 matrix $E_{21}$ subtracts 4 times row 1 from row 2? What matrix $P_{32}$ exchanges row 2 and row 3? If you multiply $A$ on the *right* instead of the left, describe the results $AE_{21}$ and $AP_{32}$.

**Solution** By doing those operations on the identity matrix $I$, we find

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying by $E_{21}$ on the right side will subtract 4 times **column 2** from **column 1**. Multiplying by $P_{32}$ on the right will exchange **columns 2** and **3**.

## II.4. Matrix Operations

$$A + B = B + A \quad \text{(commutative law)}$$
$$c(A + B) = cA + cB \quad \text{(distributive law)}$$
$$A + (B + C) = (A + B) + C \quad \text{(associative law).}$$

Three more laws hold for multiplication, but $AB = BA$ is not one of them:

$$AB \neq BA \quad \text{(the commutative "law" is } \textit{usually broken}\text{)}$$
$$C(A + B) = CA + CB \quad \text{(distributive law from the left)}$$
$$(A + B)C = AC + BC \quad \text{(distributive law from the right)}$$
$$A(BC) = (AB)C \quad \text{(associative law for } ABC\text{) (}\textbf{\textit{parentheses not needed}}\text{).}$$

When $A$ and $B$ are not square, $AB$ is a different size from $BA$. These matrices can't be equal—even if both multiplications are allowed. For square matrices, almost any example shows that $AB$ is different from $BA$:

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}.$$

### II.4.A. Block Matrices

**4 by 6 matrix**
**2 by 2 blocks**

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

If $B$ is also 4 by 6 and the block sizes match, you can add $A + B$ *a block at a time*.

We have seen block matrices before. The right side vector $\boldsymbol{b}$ was placed next to $A$ in the "augmented matrix". Then $[\,A \quad \boldsymbol{b}\,]$ has two blocks of different sizes. Multiplying by an elimination matrix gave $[\,EA \quad E\boldsymbol{b}\,]$. No problem to multiply blocks times blocks, when their shapes permit.

**Block multiplication** If the cuts between columns of $A$ match the cuts between rows of $B$, then block multiplication of $AB$ is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}. \quad (1)$$

**Block elimination** $\left[\begin{array}{c|c} I & \mathbf{0} \\ \hline -CA^{-1} & I \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline \mathbf{0} & D - CA^{-1}B \end{array}\right].$

## II.5. Inverse Matrices

**DEFINITION** The matrix $A$ is ***invertible*** if there exists a matrix $A^{-1}$ such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \tag{1}$$

***Not all matrices have inverses.*** This is the first question we ask about a square matrix: Is $A$ invertible? We don't mean that we immediately calculate $A^{-1}$. In most problems we never compute it! Here are six "notes" about $A^{-1}$.

**Note 1** ***The inverse exists if and only if elimination produces $n$ pivots*** (row exchanges are allowed). Elimination solves $A\mathbf{x} = \mathbf{b}$ without explicitly using the matrix $A^{-1}$.

**Note 2** The matrix $A$ cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this "proof by parentheses":

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \tag{2}$$

This shows that a *left-inverse* $B$ (multiplying from the left) and a *right-inverse* $C$ (multiplying $A$ from the right to give $AC = I$) must be the *same matrix*.

**Note 3** If $A$ is invertible, the one and only solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$:

***Multiply*** $A\mathbf{x} = \mathbf{b}$ ***by*** $A^{-1}$. ***Then*** $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$.

**Note 4** (Important) ***Suppose there is a nonzero vector $\mathbf{x}$ such that*** $A\mathbf{x} = \mathbf{0}$. *Then $A$ cannot have an inverse.* No matrix can bring $\mathbf{0}$ back to $\mathbf{x}$.

If $A$ is invertible, then $A\mathbf{x} = \mathbf{0}$ can only have the zero solution $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.

**Note 5** A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\textbf{2 by 2 Inverse:} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3}$$

This number $ad - bc$ is the *determinant* of $A$. A matrix is invertible if its determinant is nc zero (Chapter 5). The test for $n$ pivots is usually decided before the determinant appears.

**Example 1** The 2 by 2 matrix $A = \left[\begin{smallmatrix}1 & 2\\ 1 & 2\end{smallmatrix}\right]$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $Ax = \mathbf{0}$ when $x = (2, -1)$. It fails to have two pivots as required by Note 1.

Elimination turns the second row of this matrix $A$ into a zero row.

**Note 6** A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If } A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

$$(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$$

## II.5.A. Gauss-Jordan Elimination

$$\begin{bmatrix} K & e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \quad \textbf{Start Gauss-Jordan on } K$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & \mathbf{-1} & \frac{1}{2} & \mathbf{1} & \mathbf{0} \\ 0 & -1 & 2 & 0 & 0 & 1 \end{bmatrix} \quad (\tfrac{1}{2}\ \textit{row}\ 1 + \textit{row}\ 2)$$

$$\rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ \mathbf{0} & \mathbf{0} & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & \mathbf{1} \end{bmatrix} \quad (\tfrac{2}{3}\ \textit{row}\ 2 + \textit{row}\ 3)$$

$$\begin{pmatrix} \text{Zero above} \\ \text{third pivot} \end{pmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 0 & 1 & 0 & 0 \\ \mathbf{0} & \frac{3}{2} & \mathbf{0} & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (\tfrac{3}{4}\ \textit{row}\ 3 + \textit{row}\ 2)$$

$$\begin{pmatrix} \text{Zero above} \\ \text{second pivot} \end{pmatrix} \rightarrow \begin{bmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} & \frac{3}{2} & \mathbf{1} & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix} \quad (\tfrac{2}{3}\ \textit{row}\ 2 + \textit{row}\ 1)$$

(divide by 2)
(divide by $\frac{3}{2}$)
(divide by $\frac{4}{3}$)

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} = [\, I \;\; x_1 \;\; x_2 \;\; x_3 \,] = [\, I \;\; K^{-1} \,].$$

Starting from the 3 by 6 matrix $[\, K \;\; I \,]$, we ended with $[\, I \;\; K^{-1} \,]$. Here is the whole Gauss-Jordan process on one line for any invertible matrix $A$:

**Gauss-Jordan** ***Multiply*** $[\, A \;\; I \,]$ ***by*** $A^{-1}$ ***to get*** $[\, I \;\; A^{-1} \,]$.

1. $K$ is ***symmetric*** across its main diagonal. So is $K^{-1}$.
2. $K$ is ***tridiagonal*** (only three nonzero diagonals). But $K^{-1}$ is a dense matrix with no zeros. That is another reason we don't often compute inverse matrices. The inverse of a band matrix is generally a dense matrix.
3. The *product of pivots* is $2(\frac{3}{2})(\frac{4}{3}) = 4$. This number 4 is the ***determinant*** of $K$.

$K^{-1}$ ***involves division by the determinant***
$$K^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \tag{8}$$

**This is why an invertible matrix cannot have a zero determinant.**

A triangular matrix is invertible if and only if no diagonal entries are zero.

**Gauss-Jordan on triangular** $L$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{bmatrix} = [\, L \;\; I \,]$$

$$\begin{matrix} \\ \to \\ \to \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix}$$

(3 times row 1 from row 2)
(4 times row 1 from row 3)
(then 5 times row 2 from row 3)

$$\begin{matrix} \\ \\ \to \end{matrix}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 11 & -5 & 1 \end{bmatrix} = [\, I \;\; L^{-1} \,].$$

$A$ is invertible if and only if it has $n$ pivots (row exchanges allowed).

If $A\underline{x} = 0$ for a nonzero vector $\underline{x}$, then A has no inverse.

Consider the following set of matrices $A, B, C, D, S, E$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

**Solution**

$$B^{-1} = \frac{1}{4}\begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36}\begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

## II.6. Elimination = Factorization

Elimination matrix is LT, and after elimination $U$ is UT $\Rightarrow EA = U \Rightarrow A = E^{-1}U$.

But, $E^{-1}$ is also LT $A = LU$.

**Example 1** Elimination subtracts $\frac{1}{2}$ times row 1 from row 2. The last step subtracts $\frac{2}{3}$ times row 2 from row 3. The lower triangular $L$ has $\ell_{21} = \frac{1}{2}$ and $\ell_{32} = \frac{2}{3}$. Multiplying $LU$ produces $A$:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU.$$

The (3, 1) multiplier is zero because the (3, 1) entry in $A$ is zero. No operation needed.

$$A\underline{x} = \underline{b} \Rightarrow U\underline{x} = \underline{c} \Rightarrow \underline{c} = U\underline{x}$$
$$\underline{c} = U\underline{x} \Rightarrow L\underline{c} = LU\underline{x} = A\underline{x} = \underline{b}$$
$$L\underline{c} = \underline{b}$$

Therefore, solve $L\underline{c} = \underline{b}$ to find $\underline{c}$, then solve $U\underline{x} = \underline{c}$ to find $\underline{x}$.

**Example 3** Forward elimination (downward) on $Ax = b$ ends at $Ux = c$:

$$Ax = b \quad \begin{aligned} u + 2v &= 5 \\ 4u + 9v &= 21 \end{aligned} \quad \text{becomes} \quad \begin{aligned} u + 2v &= 5 \\ v &= 1 \end{aligned} \quad Ux = c$$

The multiplier was 4, which is saved in $L$. The right side used it to change 21 to 1:

$Lc = b$ The lower triangular system $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$ gave $c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$Ux = c$ The upper triangular system $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ gives $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

**Better balance** The $LU$ factorization is "unsymmetric" because $U$ has the pivots on its diagonal where $L$ has 1's. This is easy to change. ***Divide U by a diagonal matrix D that contains the pivots.*** That leaves a new matrix with 1's on the diagonal:

$$\text{Split } U \text{ into } \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}.$$

It is convenient (but a little confusing) to keep the same letter $U$ for this new upper triangular matrix. It has 1's on the diagonal (like $L$). Instead of the normal $LU$, the new form has $D$ in the middle: ***Lower triangular L times diagonal D times upper triangular U.***

***The triangular factorization can be written $A = LU$ or $A = LDU$.***

Whenever you see $LDU$, it is understood that $U$ has 1's on the diagonal. *Each row is divided by its first nonzero entry—the pivot.* Then $L$ and $U$ are treated evenly in $LDU$:

## II.7. Transposes and Permutations

**Transpose** If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ then $A^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$.

You can write the rows of $A$ into the columns of $A^{\mathrm{T}}$. Or you can write the columns of $A$ into the rows of $A^{\mathrm{T}}$. The matrix "flips over" its main diagonal. The entry in row $i$, column $j$ of $A^{\mathrm{T}}$ comes from row $j$, column $i$ of the original $A$:

**Exchange rows and columns** $(A^{\mathrm{T}})_{ij} = A_{ji}$.

The transpose of a lower triangular matrix is upper triangular. (But the inverse is still lower triangular.) The transpose of $A^{\mathrm{T}}$ is $A$.

**Sum** The transpose of $A + B$ is $A^{\mathrm{T}} + B^{\mathrm{T}}$. (1)

**Product** The transpose of $AB$ is $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$. (2)

**Inverse** The transpose of $A^{-1}$ is $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1}$. (3)

Notice especially how $B^{\mathrm{T}}A^{\mathrm{T}}$ comes in reverse order. For inverses, this reverse order was quick to check: $B^{-1}A^{-1}$ times $AB$ produces $I$. To understand $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, start with $(A\boldsymbol{x})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$:

*$A\boldsymbol{x}$ combines the columns of $A$ while $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}$ combines the rows of $A^{\mathrm{T}}$.*

### II.7.A. Symmetric Matrices

The inverse of a symmetric matrix is also symmetric.

$A^T A$ is always symmetric.

**Example 4** Multiply $R = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $R^{\mathrm{T}} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$ in both orders.

$RR^{\mathrm{T}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $R^{\mathrm{T}}R = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ are both symmetric matrices.

The product $R^{\mathrm{T}}R$ is $n$ by $n$. In the opposite order, $RR^{\mathrm{T}}$ is $m$ by $m$. Both are symmetric, with positive diagonal (*why*?). But even if $m = n$, it is not very likely that $R^{\mathrm{T}}R = RR^{\mathrm{T}}$. Equality can happen, but it is abnormal.

$$\begin{bmatrix}1 & 2\\ 2 & 7\end{bmatrix} = \begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix} \quad LU \text{ misses the symmetry of } A$$

$$= \begin{bmatrix}\mathbf{1} & 0\\ \mathbf{2} & \mathbf{1}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{2}\\ 0 & \mathbf{1}\end{bmatrix} \quad LDU \text{ captures the symmetry}$$

***Now U is the transpose of L.***

The symmetric factorization of a symmetric matrix is $A = LDL^T$.

## II.7.B. Permutation Matrices

**DEFINITION** *A permutation matrix P has the rows of the identity I in any order.*

**Example 5** There are six 3 by 3 permutation matrices. Here they are without the zeros:

$$I = \begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} \quad P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1\end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1\\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix}1 & & \\ & & 1\\ & 1 & \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1\\ 1 & & \\ & 1 & \end{bmatrix}.$$

$$P^{-1} = P^T$$

# III. Vector Spaces and Subspaces

The space $\mathbb{R}^n$ consists of all column vectors with $n$ components.

$$\begin{bmatrix} 4 \\ \pi \end{bmatrix} \in \mathbb{R}^2, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^5, \quad \begin{bmatrix} 1-i \\ 2+3i \end{bmatrix} \in \mathbb{C}^2$$

(1) $x + y = y + x$

(2) $x + (y + z) = (x + y) + z$

(3) There is a unique "zero vector" such that $x + \mathbf{0} = x$ for all $x$

(4) For each $x$ there is a unique vector $-x$ such that $x + (-x) = \mathbf{0}$

(5) 1 times $x$ equals $x$

(6) $(c_1 c_2)x = c_1(c_2 x)$

(7) $c(x + y) = cx + cy$

(8) $(c_1 + c_2)x = c_1 x + c_2 x$.

$M$ The vector space of all real 2 by 2 matrices.

$Z$ The vector space that consists only of a zero vector.

## III.1.A. Subspaces

A plane through the origin (0,0,0) is a subspace of $\mathbb{R}^3$.

**DEFINITION** A *subspace* of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: ***If $v$ and $w$ are vectors in the subspace and $c$ is any scalar, then***

(i) $v + w$ is in the subspace

(ii) $cv$ is in the subspace.

Every subspace contains the zero vector (point of the origin).

Lines through the origin are subspaces.

$\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

**Example 1** Keep only the vectors $(x, y)$ whose components are positive or zero (this is a quarter-plane). The vector $(2, 3)$ is included but $(-2, -3)$ is not. So rule (**ii**) is violated when we try to multiply by $c = -1$. ***The quarter-plane is not a subspace.***

**Example 2** Include also the vectors whose components are both negative. Now we have two quarter-planes. Requirement (**ii**) is satisfied; we can multiply by any $c$. But rule (**i**) now fails. The sum of $\boldsymbol{v} = (2, 3)$ and $\boldsymbol{w} = (-3, -2)$ is $(-1, 1)$, which is outside the quarter-planes. ***Two quarter-planes don't make a subspace.***

Rules (**i**) and (**ii**) involve vector addition $\boldsymbol{v} + \boldsymbol{w}$ and multiplication by scalars like $c$ and $d$. The rules can be combined into a single requirement—*the rule for subspaces*:

***A subspace containing $\boldsymbol{v}$ and $\boldsymbol{w}$ must contain all linear combinations $c\boldsymbol{v} + d\boldsymbol{w}$.***

**Example 3** Inside the vector space **M** of all 2 by 2 matrices, here are two subspaces:

(**U**) All upper triangular matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ (**D**) All diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

## III.1.B. Column Space

$C(A)$ contains all columns of $A$ and all their linear combinations $A\underline{x}$.

The system $A\underline{x} = \underline{b}$ is solvable if and only if $\underline{b}$ is in the column space of $A$.

Let $A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \Rightarrow A\underline{x} = \underline{b}$ means $\underline{b} = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \Rightarrow$ plane in $\mathbb{R}^3$.

**Important** Instead of columns in $\mathbf{R}^m$, we could start with any set **S** of vectors in a vector space **V**. To get a subspace **SS** of **V**, we take *all combinations* of the vectors in that set:

$$\mathbf{S} = \text{set of vectors in } \mathbf{V} \text{ (probably } \textit{not} \text{ a subspace)}$$
$$\mathbf{SS} = \text{all combinations of vectors in } \mathbf{S}$$

$$\mathbf{SS} = \text{all } c_1\boldsymbol{v}_1 + \cdots + c_N\boldsymbol{v}_N = \textbf{the subspace of V “spanned” by S}$$

When **S** is the set of columns, **SS** is the column space. When there is only one nonzero vector $\boldsymbol{v}$ in **S**, the subspace **SS** is the line through $\boldsymbol{v}$. *Always* **SS** *is the smallest subspace containing* **S**. This is a fundamental way to create subspaces and we will come back to it.

**The subspace SS is the “span” of S, containing all combinations of vectors in S.**

Find $C(A)$ for

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

## III.2. The Nullspace

Vectors $\underline{x}$ that are solutions to $A\underline{x} = 0$ are in the null space of $A$.

**Example 1** $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = [1 \quad 2 \quad 3]$. This equation $A\mathbf{x} = \mathbf{0}$ produces a plane through the origin $(0, 0, 0)$. The plane is a subspace of $\mathbf{R}^3$. *It is the nullspace of $A$.*

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

**Example 2** Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. This matrix is singular!

**Solution** Apply elimination to the linear equations $A\mathbf{x} = \mathbf{0}$:

$$\begin{matrix} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{matrix} \quad \rightarrow \quad \begin{matrix} x_1 + 2x_2 = 0 \\ \mathbf{0} = \mathbf{0} \end{matrix}$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $\mathbf{N}(A)$. It contains all solutions $(x_1, x_2)$.

To describe this line of solutions, here is an efficient way. Choose one point on the line (one "***special solution***"). Then all points on the line are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. The special solution $\boldsymbol{s}$ is $(-2, 1)$:

**Special solution** The nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ contains all multiples of $\boldsymbol{s} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

What is the null space of $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$?

***The nullspace consists of all combinations of the special solutions.***

The plane $x + 2y + 3z = 0$ in Example 1 had *two* special solutions:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \text{ has the special solutions } s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}$$

$N(A) = Z, N(B) = Z$ .

$$C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

↑ ↑ ↑ ↑

**pivot columns free columns**

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{pivot} \\ \leftarrow \text{variables} \\ \leftarrow \text{free} \\ \leftarrow \text{variables} \end{matrix}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular $U$! We can continue to make this matrix simpler, in two ways:

1. *Produce zeros above the pivots,* **by eliminating upward.**
2. *Produce ones in the pivots,* **by dividing the whole row by its pivot.**

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same. This nullspace becomes easiest to see when we reach the ***reduced row echelon form*** $R$. It has $I$ in the pivot columns:

**Reduced form $R$**

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

↑ ↑

now the pivot columns contain $I$

Let $A=\begin{bmatrix}1 & 1 & 2 & 3\\ 2 & 2 & 8 & 10\\ 3 & 3 & 10 & 13\end{bmatrix}\Rightarrow$

**Triangular $U$ :** $U=\begin{bmatrix}\mathbf{1} & 1 & 2 & 3\\ 0 & 0 & \mathbf{4} & 4\\ 0 & 0 & 0 & 0\end{bmatrix}$ *Only two pivots* *The last equation became* $0=0$

Pivot columns: 1,3

Free columns: 2,4

| | | |
|---|---|---|
| **P** | The *pivot* variables are $x_1$ and $x_3$. | **Columns 1 and 3 contain pivots.** |
| **F** | The *free* variables are $x_2$ and $x_4$. | **Columns 2 and 4 have no pivots.** |

***Special solutions*** to $x_1+x_2+2x_3+3x_4=0$ and $4x_3+4x_4=0$

- Set $x_2=1$ and $x_4=0$. By back substitution $x_3=0$. Then $x_1=-1$.
- Set $x_2=0$ and $x_4=1$. By back substitution $x_3=-1$. Then $x_1=-1$.

These special solutions solve $U\boldsymbol{x}=\mathbf{0}$ and therefore $A\boldsymbol{x}=\mathbf{0}$. They are in the nullspace. The good thing is that *every solution is a combination of the special solutions.*

***Complete solution to*** $A\boldsymbol{x}=\mathbf{0}$

$$\boldsymbol{x}=x_2\underset{\textbf{special}}{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}}+x_4\underset{\textbf{special}}{\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}}=\underset{\textbf{complete}}{\begin{bmatrix}-x_2-x_4\\x_2\\-x_4\\x_4\end{bmatrix}}. \quad (1)$$

The special solutions are in the nullspace, and their combinations fill out the whole nullspace.

**Example 4** Find the nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$.

The second column of $U$ has no pivot. So $x_2$ is free. The special solution has $x_2 = 1$. Back substitution into $9x_3 = 0$ gives $x_3 = 0$. Then $x_1 + 5x_2 = 0$ or $x_1 = -5$. The solutions to $U\boldsymbol{x} = \boldsymbol{0}$ are multiples of one special solution:

$$x = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix}$$

The nullspace of $U$ is a line in $\mathbf{R}^3$.
It contains multiples of the special solution $s = (-5, 1, 0)$.
One variable is free, and $N =$ **nullbasis** $(U)$ has one column $s$.

## III.2.A. ECHELON MATRICES

$$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{rref}(U).$$

This makes it even clearer that the special solution (column of $N$) is $s = (-5, 1, 0)$.

$$U = \begin{bmatrix} p & x & x & x & x & x & x \\ 0 & p & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & p & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Three pivot variables $x_1, x_2, x_6$**
**Four free variables $x_3, x_4, x_5, x_7$**
**Four special solutions in $N(U)$**

Suppose $A\boldsymbol{x} = \boldsymbol{0}$ has more unknowns than equations ($\mathbf{n} > \mathbf{m}$, more columns than rows). Then there are **nonzero solutions**. There must be free columns, without pivots.

## III.2.B. ROW-REDUCED ECHELON MATRICES

From an echelon matrix $U$ we go one more step. Continue with a 3 by 4 example:

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

*We can divide the second row by* 4. Then both pivots equal 1. *We can subtract* 2 *times this new row* $\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$ *from the row above.* ***The reduced row echelon matrix R has zeros above the pivots as well as below:***

**Reduced row echelon matrix**

$$R = \text{rref}(A) = \begin{bmatrix} \mathbf{1} & 1 & \mathbf{0} & 1 \\ \mathbf{0} & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Pivot rows contain I**

***R has 1's as pivots. Zeros above pivots come from upward elimination.***

**1.** Set $x_2 = 1$ and $x_4 = 0$. Solve $Rx = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = 0$.
Those numbers $-1$ and $0$ are sitting in column 2 of $R$ (with plus signs).

**2.** Set $x_2 = 0$ and $x_4 = 1$. Solve $Rx = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = -1$.
Those numbers $-1$ and $-1$ are sitting in column 4 (with plus signs).

*By reversing signs we can read off the special solutions directly from $R$. The nullspace $N(A) = N(U) = N(R)$ contains all combinations of the special solutions:*

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (\textit{complete solution of } Ax = \mathbf{0}).$$

**3.2 A** Create a 3 by 4 matrix whose special solutions to $Ax = \mathbf{0}$ are $s_1$ and $s_2$:

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

pivot columns 1 and 3
free variables $x_2$ and $x_4$

You could create the matrix $A$ in row reduced form $R$. Then describe all possible matrices $A$ with the required nullspace $N(A)$ = all combinations of $s_1$ and $s_2$.

**Solution** The reduced matrix $R$ has pivots = 1 in columns 1 and 3. There is no third pivot, so the third row of $R$ is all zeros. The free columns 2 and 4 will be combinations of the pivot columns:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad Rs_1 = \mathbf{0} \quad \text{and} \quad Rs_2 = \mathbf{0}.$$

The entries 3, 2, 6 in $R$ are the negatives of $-3, -2, -6$ in the special solutions!

$R$ is only one matrix (one possible $A$) with the required nullspace. We could do any elementary operations on $R$—exchange rows, multiply a row by any $c \neq 0$, subtract any multiple of one row from another. **$R$ can be multiplied** (*on the left*) **by any invertible matrix, without changing its nullspace.**

Every 3 by 4 matrix has at least one special solution. *These matrices have two.*

## III.3. The Rank and the Row Reduced Form

Rank $r$ = number of pivots.

$r(A) = r(U) = r(R)$.

**Rank one matrix** $A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \longrightarrow R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The column space of a rank one matrix is "one-dimensional". Here all columns are on the line through $\boldsymbol{u} = (1, 2, 3)$. The columns of $A$ are $\boldsymbol{u}$ and $3\boldsymbol{u}$ and $10\boldsymbol{u}$. Put those numbers into the row $\boldsymbol{v}^{\mathrm{T}} = [\,1 \quad 3 \quad 10\,]$ and you have the special rank one form $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$:

$$A = \textbf{column times row} = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \quad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \quad 3 \quad 10] \qquad (3)$$

Note that $A\underline{x} = 0 \Rightarrow \underline{u}\underline{v}^T\underline{x} = 0$. Note that $\underline{v}^T\underline{x}$ is a scalar, $\Rightarrow \underline{v}^T\underline{x} = 0$.

This means that vectors in the null space of $A$ are orthogonal to $\underline{v}$.

Note that $\underline{v}$ is in the row space of $A$.

**Pivot row** $[1 \;\; 3 \;\; 10]$
**Pivot variable** $x_1$
**Free variables** $x_2$ **and** $x_3$

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \qquad s_2 = \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}$$

The nullspace contains all combinations of $s_1$ and $s_2$. This produces the plane $x + 3y + 10z = 0$, perpendicular to the row $(1, 3, 10)$. **Nullspace** (plane) **perpendicular to row space** (line).

Rank = number of independent columns = number of independent rows.

Rank = dimension of the column space = dimension of the row space.

## III.3.A. The Pivot Columns

Pivot columns of $A, U, R$ are the same. Column spaces are different.

Pivot columns of $R$ form an $r \times r$ identity matrix, sitting above $m - r$ rows of zeros.

## III.3.B. The Special Solutions

Columns of null space matrix are the special solutions.

$AN = 0$

$N$ is $n \times (n - r)$.

The special solutions are easy for $Rx = \mathbf{0}$. Suppose that the first $r$ columns are the pivot columns. Then the reduced row echelon form looks like

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{matrix} r \text{ pivot rows} \\ m-r \text{ zero rows} \end{matrix} \qquad (4)$$

$r$ pivot columns $\quad n-r$ free columns

The pivot variables in the $n-r$ special solutions come by changing $F$ to $-F$:

**Nullspace matrix** $\quad N = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{matrix} r \text{ pivot variables} \\ n-r \text{ free variables} \end{matrix} \qquad (5)$

**Example 2** The special solutions of $Rx = x_1 + 2x_2 + 3x_3 = 0$ are the columns of $N$:

$$R = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad N = \begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The rank is one. There are $n - r = 3 - 1$ special solutions $(-2, 1, 0)$ and $(-3, 0, 1)$.

## III.4. The Complete Solution

$A\underline{x} = \underline{b} \Rightarrow R\underline{x} = \underline{d}$

Let $\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{bmatrix} = \begin{bmatrix} A & \underline{b} \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} \underline{1} & 3 & 0 & 2 & b_1 \\ 0 & 0 & \underline{1} & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{bmatrix} = \begin{bmatrix} R & \underline{d} \end{bmatrix}$$

Then from the last row, the equations are consistent if $0 = b_3 - b_1 - b_2 \Rightarrow b_1 + b_2 = b_3$.

Let $\underline{b} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 3 & 0 & 2 & 1 \\ 0 & 0 & \underline{1} & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R & \underline{d} \end{bmatrix} \Rightarrow \underline{d} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$

Pivot variables $x_1, x_3$. Free variables $x_2, x_4$.

Particular solution: $x_2 = x_4 = 0 \Rightarrow x_1 = 1, x_3 = 6$. Free variables are zeros and pivot variables are from $\underline{d}$.

$$\underline{x}_p = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

Note that $R\underline{x}_p = \underline{d}$ is satisfied.

$$\begin{matrix} A\underline{x}_p = \underline{b} \\ A\underline{x}_n = 0 \end{matrix} \Rightarrow A\left(\underline{x}_p + \underline{x}_n\right) = \underline{b}$$

$$\underline{x} = \underline{x}_p + \underline{x}_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Let $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \underline{x} = \underline{b}$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 1 & b_1 \\ 0 & \underline{1} & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 0 & 2b_1 - b_2 \\ 0 & \underline{1} & b_2 - b_1 \\ 0 & 0 & b_3 + b_2 + b_1 \end{bmatrix}$$

The system is solvable when $b_3 + b_2 + b_1 = 0$.

Two pivot variables, no free variables $\Rightarrow r = 2$. No special solutions.

$\underline{x} = \underline{x}_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$ This is only one solution.

$A$ has full column rank. $R = \begin{bmatrix} I \\ m - n \text{ rows of zeros} \end{bmatrix}$. $\mathcal{N}(A) = Z$.

Row space of $A$ is $C\left(A^T\right)$.

## III.5. Independence, Basis and Dimension

Basis vectors:

1. Independent
2. Span the space

There is one and only one way to write a vector as a combination of the basis vectors.

The basis is not unique.

The columns of every invertible $n \times n$ matrix give a basis for $\mathbb{R}^n$.

The pivot columns of $A$ are a basis for its column space. The pivot rows of $A$ are a basis for its row space. So are the pivot rows of its echelon form $R$.

Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

Basis of column space = $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Basis of row space = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Set of basis vectors is not unique.

If $\left\{v_i\right\}_{i=1}^{m}$ and $\left\{w_i\right\}_{i=1}^{n}$ are both sets of basis vectors, then $n = m$.

Dimension of a space is the number of basis vectors.

Dimension of column space=rank of matrix.

Space $M$ contains all $2 \times 2$ matrices. One basis is:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Note that $c_1 A_1 + c_2 A_2 + c_3 A_3 + c_4 A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = A$=arbitrary matrix in $M$.

Note that $A=0$ only if $c_1=c_2=c_3=c_4=0$.

$M$ is 4-dimensional.

$A_1, A_2, A_4$ are basis of the 3-dimansional space of upper triangular matrices.

**3.5 A** Start with the vectors $v_1=(1,2,0)$ and $v_2=(2,3,0)$. **(a)** Are they linearly independent? **(b)** Are they a basis for any space? **(c)** What space **V** do they span? **(d)** What is the dimension of **V**? **(e)** Which matrices $A$ have **V** as their column space? **(f)** Which matrices have **V** as their nullspace? **(g)** Describe all vectors $v_3$ that complete a basis $v_1, v_2, v_3$ for $\mathbf{R}^3$.

**Solution**

**(a)** $v_1$ and $v_2$ are independent—the only combination to give $\mathbf{0}$ is $0v_1+0v_2$.

**(b)** Yes, they are a basis for the space they span.

**(c)** That space **V** contains all vectors $(x,y,0)$. It is the $xy$ plane in $\mathbf{R}^3$.

**(d)** The dimension of **V** is 2 since the basis contains two vectors.

**(e)** This **V** is the column space of any 3 by $n$ matrix $A$ of rank 2, if every column is a combination of $v_1$ and $v_2$. In particular $A$ could just have columns $v_1$ and $v_2$.

**(f)** This **V** is the nullspace of any $m$ by 3 matrix $B$ of rank 1, if every row is a multiple of $(0,0,1)$. In particular take $B=[0\ 0\ 1]$. Then $Bv_1=\mathbf{0}$ and $Bv_2=\mathbf{0}$.

**(g)** Any third vector $v_3=(a,b,c)$ will complete a basis for $\mathbf{R}^3$ provided $c \neq 0$.

$A$ is $m \times n$.

***Four Fundamental Subspaces***

The ***row space*** is $C(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^n$.

The ***column space*** is $C(A)$, a subspace of $\mathbf{R}^m$.

The ***nullspace*** is $N(A)$, a subspace of $\mathbf{R}^n$.

The ***left nullspace*** is $N(A^{\mathrm{T}})$, a subspace of $\mathbf{R}^m$. This is our new space.

Left Null Space = $\mathcal{N}\left(A^T\right)$. Solve $A^T \underline{y} = 0$. $A^T$ is $n \times m$.

| Fundamental Theorem of Linear Algebra – Part I: |
| --- |
| $\mathcal{N}\left(A^T\right)$ is a subspace of $\mathbb{R}^m$. Dimension $m-r$. |
| $\mathcal{N}(A)$ is a subspace of $\mathbb{R}^n$. Dimension $n-r$. |
| The row space and column space have the same dimension $r$. |

Let $A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Pivot rows: 1,2 – Pivot columns: 1,4 - $r = 2$.

Rows space dimension = 2 – Pivot rows are basis.

Column space dimension = 2 – Pivot columns are basis.

Null space dimension = 3 – Special solutions are basis.

Left null space dimension = 1.

$R^T \underline{y} = 0 \Rightarrow \underline{y}^T R = 0 \Rightarrow$ This means that $\underline{y}^T$ is a linear combination of the rows that is zero.

Pivot rows of $R$ are independent – their combination should have zero coefficients.

$\underline{y}^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow \underline{y}$ is a basis for the left null space. Last $m-r$ rows of $I$.

The number of independent columns equals the number of independent rows.

$A$ and $R$ have same row space.

$C(A) \neq C(R)$. Pivot columns of $A$ are basis for $C(A)$.

$A$ and $R$ have same null space.

# IV. ORTHOGONALITY

Right triangle: $a^2 + b^2 = c^2$

$$\underline{v}^T \underline{w} = 0 \Rightarrow$$
$$\|\underline{v} + \underline{w}\|^2 = (\underline{v} + \underline{w})^T (\underline{v} + \underline{w}) = \|\underline{v}\|^2 + 2\underline{v}^T \underline{w} + \|\underline{w}\|^2 \Rightarrow$$
$$\|\underline{v} + \underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2$$

Every row of $A$ is perpendicular to every solution of $A\underline{x} = 0 \Rightarrow$

**The row space is perpendicular to the null space.**

Every column of $A$ is perpendicular to every solution of $\underline{y}^T A = 0 \Rightarrow$

**The column space is perpendicular to the left null space.**

Spaces $V$ and $W$ are orthogonal ($V \perp W$) if for every $\underline{v} \in V$ and every $\underline{w} \in W$, $\underline{v}^T \underline{w} = 0$.

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix}, \underline{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow A\underline{x} = 0$.

$\underline{x} \in \mathcal{N}(A)$. Rows of $A$ are perpendicular to $\underline{x}$.

Row space and null space are orthogonal complements because their dimensions add to $n$.

| |
|---|
| Fundamental Theorem of Linear Algebra – Part II: |
| $\mathcal{N}(A)$ is the orthogonal complement of $C(A^T)$. |
| $\mathcal{N}(A^T)$ is the orthogonal complement of $C(A)$. |

Let $\underline{x}$ be split into a row space component and a null space component, $\underline{x} = \underline{x}_n + \underline{x}_r$.

$$\begin{aligned} A\underline{x} &= A\underline{x}_n + A\underline{x}_r \\ &= A\underline{x}_r \end{aligned}$$

Vectors in the row space can be transformed into vectors in the column space by $A\underline{x}$.

Let $A=\begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 \\ 0 & 0 & A_{33} & 0 & 0 \end{bmatrix}$, this is a diagonal matrix.

If $A_{11}, A_{22}, A_{33} \neq 0 \Rightarrow r = 3 \Rightarrow$

$A$ includes the $3\times 3$ invertible diagonal submatrix $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}$

Let $A=\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r = 2 \Rightarrow$ Submatrix=$\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

Let $B=\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \underline{1} & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \underline{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \underline{1} & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$

Pivot cols: 1,3 – Submatrix=$\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$.

Let $A=\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\underline{x}=\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Since $\underline{x}_r = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ is in the row space, then $\underline{x}_n = \underline{x} - \underline{x}_r = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in the null space.

Note that $\underline{x}_r \perp \underline{x}_n$.

**4.1 A** Suppose $S$ is a six-dimensional subspace of nine-dimensional space $\mathbf{R}^9$.

(a) What are the possible dimensions of subspaces orthogonal to $S$?

(b) What are the possible dimensions of the orthogonal complement $S^\perp$ of $S$?

(c) What is the smallest possible size of a matrix $A$ that has row space $S$?

(d) What is the shape of its nullspace matrix $N$?

**Solution**

(a) If $S$ is six-dimensional in $\mathbf{R}^9$, subspaces orthogonal to $S$ can have dimensions 0, 1, 2, 3.

(b) The complement $S^\perp$ is the largest orthogonal subspace, with dimension 3.

(c) The smallest matrix $A$ is 6 by 9 (its six rows are a basis for $S$).

(d) Its nullspace matrix $N$ is 9 by 3. The columns of $N$ contain a basis for $S^\perp$.

If a new row 7 of $B$ is a combination of the six rows of $A$, then $B$ has the same row space as $A$. It also has the same nullspace matrix $N$. The special solutions $s_1, s_2, s_3$ will be the same. Elimination will change row 7 of $B$ to all zeros.

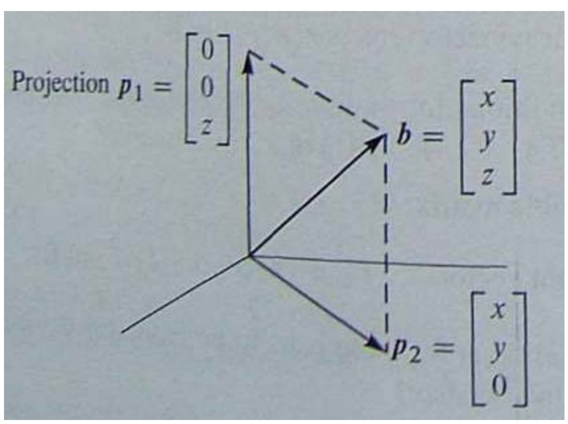
## IV.1. Projections

$\underline{p} = P\underline{b}$ is a projection of $\underline{b}$.

Note that $\underline{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ corresponds to a points in the xyz space.

When $\underline{b}$ is projected onto a line, the projection is the part of $\underline{b}$ along that line.

When $\underline{b}$ is projected onto a plane, the projection is the part of $\underline{b}$ along that plane.




$\underline{p}_1$ : projection onto z axis.

$\underline{p}_2$ : projection onto xy plane.

$\underline{p}_1 + \underline{p}_2 = \underline{b}$

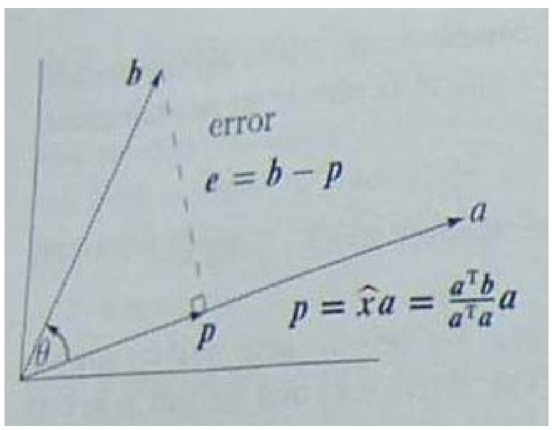
Note that $\underline{p}_1 \perp \underline{p}_2$.

Note that $\underline{p}_1$ has only the third row of $\underline{b}$, while $\underline{p}_2$ has the first two rows.

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$P_1 + P_2 = I$ .

## IV.2. Projection Onto a Line




Determine the point $p$ on $\underline{a}$ that is closest to $\underline{b}$. The line from $\underline{b}$ to $p$ is perpendicular to the vector $\underline{a}$. This is $\underline{e}$.

Note that we have a vector $\underline{p}$ that is a multiple of $\underline{a}$. Let $\underline{p} = \hat{x}\underline{a}$. $\underline{e} = \underline{b} - \hat{x}\underline{a}$. $\underline{e} \perp \underline{a}$.

$$\underline{a} \cdot (\underline{b} - \hat{x}\underline{a}) = 0 \Rightarrow \hat{x}\|\underline{a}\|^2 = \underline{a} \cdot \underline{b} \Rightarrow \hat{x} = \frac{\underline{a} \cdot \underline{b}}{\|\underline{a}\|^2} = \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}}.$$

$$\underline{p} = \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \underline{a} \Rightarrow \|\underline{p}\| = \frac{\|\underline{a}\|\|\underline{b}\|\cos\theta}{\|\underline{a}\|^2} \|\underline{a}\| = \|\underline{b}\|\cos\theta.$$

$$\underline{p} = \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} \underline{a} = \underline{a} \frac{\underline{a}^T \underline{b}}{\underline{a}^T \underline{a}} = \frac{\underline{a}\underline{a}^T}{\underline{a}^T \underline{a}} \underline{b} = P\underline{b} \Rightarrow$$

$$P = \frac{\underline{a}\underline{a}^T}{\underline{a}^T \underline{a}}$$

$P^2 = P$ : Projecting a second time doesn't change anything.

$(I - P)\underline{b} = \underline{b} - P\underline{b} \Rightarrow I - P$ is a projection matrix too.

## IV.3. Projection Onto a Subspace

Let $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n \in \mathbb{R}^m$. Let $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n$ be independent. $\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n$ span an $n$-dimensional SS of $\mathbb{R}^m$.

Let $\underline{p} = \hat{x}_1 \underline{a}_1 + \hat{x}_2 \underline{a}_2 + \cdots + \hat{x}_n \underline{a}_n$; this is a vector in the SS.

Find $\underline{p}$ that is closest to $\underline{b}$.

Note that $\underline{p} = A\underline{\hat{x}}$, where $A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \cdots & \underline{a}_n \end{bmatrix}$, $\underline{\hat{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix}$.

Let $\underline{e} = \underline{b} - A\underline{\hat{x}}$. $\underline{e} \perp \text{SS} \Rightarrow \underline{e} \perp \underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n$.

The line and plane are orthogonal complements.

Note that the z axis is $C(P_1)$, while the xy plane is $C(P_2)$.

$$\begin{bmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_n^T \end{bmatrix} \left[ \underline{b} - A\underline{\hat{x}} \right] = 0 \Rightarrow A^T (\underline{b} - A\underline{\hat{x}}) = 0 \Rightarrow A^T A \underline{\hat{x}} = A^T \underline{b} .$$

$$\underline{\hat{x}} = (A^T A)^{-1} A^T \underline{b} \Rightarrow \underline{p} = A(A^T A)^{-1} A^T \underline{b} = P\underline{b} \Rightarrow P = A(A^T A)^{-1} A^T .$$

SS is the column space of $A$.

The error vector $\underline{b} - A\underline{\hat{x}}$ is perpendicular to that column space.

Therefore $\underline{b} - A\hat{\underline{x}}$ is in the null space of $A^T$. This means that $A^T(\underline{b} - A\hat{\underline{x}}) = 0$.

The vector $\underline{b}$ is being split into the projection $\underline{p}$ and the error $\underline{b} - \underline{p} = \underline{e}$. $\underline{p} \perp \underline{e}$.

$A^T A$ is invertible if $A$ has linearly independent columns. In this case $A^T A$ is symmetric, square ($n \times n$)and invertible.

## IV.4. Least Squares Approximations

When $A$ has more rows than columns, $A\underline{x} = \underline{b}$ can have no solution.

When there is no solution, $\underline{b} \notin C(A)$, $\underline{e} = \underline{b} - A\underline{x} \neq 0$.

Some equations may be inconsistent because of measurement noise.

We try to make $\underline{e}$ as small as possible (in terms of its squared length) using a least squares solution $\hat{\underline{x}}$.

$\underline{b}$ can be split into a part $\underline{p} \in C(A)$ and a part $\underline{e} \in N\left(A^T\right)$.

Note that even if $A\underline{x} = \underline{b}$ cannot be solved, $A\hat{\underline{x}} = \underline{p}$ can be solved.

$A\underline{x} - \underline{b} = A\underline{x} - \underline{p} - \underline{e}$. $\left(A\underline{x} - \underline{p}\right) \in C(A) \Rightarrow \underline{e} \perp \left(A\underline{x} - \underline{p}\right) \Rightarrow$

$\|A\underline{x} - \underline{b}\|^2 = \|A\underline{x} - \underline{p}\|^2 + \|\underline{e}\|^2$.

$A^T A\hat{\underline{x}} = A^T\underline{b}$.

**<u>Fitting a straight line to $m$ points:</u>**

Let's fit the points ($t,b$): (0,6), (1,1), (2,3) to a straight line.

Let the line equation be $b = tx_1 + x_2$. Unknowns: $x_1, x_2$.

$$\begin{aligned} x_2 &= 6 \\ x_1 + x_2 &= 1 \\ 2x_1 + x_2 &= 3 \end{aligned} \quad .$$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$, $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\underline{b} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}$, $A\underline{x} = \underline{b}$ has no solution.

$$\underline{\hat{x}} = (A^T A)^{-1} A^T \underline{b}$$

$$= \left( \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 10 \end{bmatrix}.$$

$$= \frac{1}{6} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -9 \\ 29 \end{bmatrix}$$

$$\underline{e} = \underline{b} - A\underline{\hat{x}}$$

$$= \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ 29 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 29 \\ 20 \\ 11 \end{bmatrix}.$$

$$= \frac{7}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Note that $\underline{e} \perp \underline{a}_1, \underline{e} \perp \underline{a}_2 \Rightarrow \begin{matrix} e_2 + 2e_3 = 0 \\ e_1 + e_2 + e_3 = 0 \end{matrix}$.

In general,

$$A = \begin{bmatrix} a_{11} & 1 \\ a_{21} & 1 \\ \vdots & \vdots \\ a_{m1} & 1 \end{bmatrix}, \ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \ A\underline{x} = \underline{b} \text{ has no solution.}$$

$$A^T A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_{11} & 1 \\ a_{21} & 1 \\ \vdots & \vdots \\ a_{m1} & 1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{i1}^2 & \sum_{i=1}^{m} a_{i1} \\ \sum_{i=1}^{m} a_{i1} & m \end{bmatrix}.$$

$$A^T \underline{b} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{i1} b_i \\ \sum_{i=1}^{m} b_i \end{bmatrix}.$$

$$\begin{bmatrix} \sum_{i=1}^{m} a_{i1}^2 & \sum_{i=1}^{m} a_{i1} \\ \sum_{i=1}^{m} a_{i1} & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{i1} b_i \\ \sum_{i=1}^{m} b_i \end{bmatrix}.$$

When the points $(t, b)$ are such that $\sum_{i=1}^{m} t_i = 0$, the columns of $A$ are orthogonal.

Note that in the above, $a_{i1} = t_i$, therefore, $A^T A$ is diagonal and is equal to $\begin{bmatrix} \sum_{i=1}^{m} t_i^2 & 0 \\ 0 & m \end{bmatrix}$.

## IV.5. Fitting by a Parabola

$b = t^2 x_1 + t x_2 + x_3$.

$$\begin{aligned} t_1^2 x_1 + t_1 x_2 + x_3 &= b_1 \\ t_2^2 x_1 + t_2 x_2 + x_3 &= b_2 \\ &\vdots \\ t_m^2 x_1 + t_m x_2 + x_3 &= b_m \end{aligned}$$

$$A = \begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ \vdots & \vdots & \vdots \\ t_m^2 & t_m & 1 \end{bmatrix},\ \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \underline{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

$A\underline{x} = \underline{b}$ has no solution. Solve $A^T A \hat{\underline{x}} = A^T \underline{b}$.

## IV.6. Orthogonal Bases and Gram-Schmidt

$\{\underline{q}_i\}_{i=1}^{n}$ are orthonormal if $\underline{q}_i^T \underline{q}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$.

Let $Q = [\underline{q}_1 \ \ \underline{q}_2 \ \ \cdots \ \ \underline{q}_n] \Rightarrow Q^T Q = I$.

When $Q$ is square, $Q^{-1} = Q^T$. When $\{\underline{q}_i\}_{i=1}^{n}$ are only orthogonal, $Q^T Q$ is diagonal.

$Q^T Q = I$: when $Q$ is rectangular, $Q^T$ is the left inverse.

The rows of a square $Q$ are orthonormal. Square $Q$ is called orthogonal.

$Q=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$ rotates every vector in the plane clockwise by an angle $\theta$. $Q$ is a rotation matrix.

$Q^{-1}=Q^T=\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$. Columns of $Q$ are orthonormal basis of $\mathbb{R}^2$.

$P=\begin{bmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0\end{bmatrix}$ is a permutation matrix. $P$ is orthogonal. $P^{-1}=P^T$.

Let $\underline{u}$ be a unit vector. $Q=I-2\underline{u}\underline{u}^T$. $Q^TQ=I$. $Q$ is symmetric and orthogonal. See examples in book.

Rotation, reflection and permutation ($Q$) preserve vector length. $\|Q\underline{x}\|=\|\underline{x}\|$.

$Q$ preserves dot products: $(Q\underline{x})\cdot(Q\underline{y})=\underline{x}\cdot\underline{y}$.

Suppose we have the system $Q\underline{x}=\underline{b}$. If this has no solution, we have to solve $Q^TQ\hat{\underline{x}}=Q^T\underline{b}\Rightarrow\hat{\underline{x}}=Q^T\underline{b}\Rightarrow\underline{p}=QQ^T\underline{b}\Rightarrow P=QQ^T$.

$$\underline{p}=\begin{bmatrix}\underline{q}_1 & \underline{q}_2 & \cdots & \underline{q}_n\end{bmatrix}\begin{bmatrix}\underline{q}_1^T\underline{b} \\ \underline{q}_2^T\underline{b} \\ \vdots \\ \underline{q}_n^T\underline{b}\end{bmatrix}=\left(\underline{q}_1^T\underline{b}\right)\underline{q}_1+\left(\underline{q}_2^T\underline{b}\right)\underline{q}_2+\cdots+\left(\underline{q}_n^T\underline{b}\right)\underline{q}_n.$$

The above has $n$ 1-dimensional projections.

When $Q$ is square,

$$\underline{b}=\underline{p}=\left(\underline{q}_1^T\underline{b}\right)\underline{q}_1+\left(\underline{q}_2^T\underline{b}\right)\underline{q}_2+\cdots+\left(\underline{q}_n^T\underline{b}\right)\underline{q}_n.$$

## IV.7. The Gram-Schmidt Process

Start with three independent vectors $\underline{a}_1,\underline{a}_2,\underline{a}_3$.

Let $\underline{u}_1 = \underline{a}_1$. $\underline{q}_1 = \frac{\underline{u}_1}{\|\underline{u}_1\|} = \frac{\underline{a}_1}{\|\underline{a}_1\|}$. We must have $\underline{u}_2 \perp \underline{u}_1$.

Project $\underline{a}_2$ onto $\underline{u}_1$ to get $p_{21} = \frac{\underline{u}_1^T \underline{a}_2}{\underline{u}_1^T \underline{u}_1}$, then set

$\underline{u}_2 = \underline{a}_2 - p_{21}\underline{u}_1$. $\underline{u}_2 \perp \underline{u}_1$. $\underline{q}_2 = \frac{\underline{u}_2}{\|\underline{u}_2\|}$. $\underline{q}_2 \perp \underline{q}_1$.

Project $\underline{a}_3$ onto $\underline{u}_1$ to get $p_{31} = \frac{\underline{u}_1^T \underline{a}_3}{\underline{u}_1^T \underline{u}_1}$, and $\underline{a}_3$ onto $\underline{u}_2$ to get $p_{32} = \frac{\underline{u}_2^T \underline{a}_3}{\underline{u}_2^T \underline{u}_2}$,then set

$\underline{u}_3 = \underline{a}_3 - p_{31}\underline{u}_1 - p_{32}\underline{u}_2$. $\underline{u}_3 \perp \underline{u}_1$, $\underline{u}_3 \perp \underline{u}_2$. $\underline{q}_3 = \frac{\underline{u}_3}{\|\underline{u}_3\|}$. $\underline{q}_3 \perp \underline{q}_1$. $\underline{q}_3 \perp \underline{q}_2$.

Let $\underline{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \underline{a}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}, \underline{a}_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$

$$\underline{u}_1 = \underline{a}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ \underline{q}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$p_{21} = 1, \underline{u}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},\ \underline{q}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$p_{31} = 3,\ p_{32} = -1,\ \underline{u}_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\ \underline{q}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Note that $A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix} \Rightarrow Q = \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \end{bmatrix} \Rightarrow A = QR$.

Let $A = QR \Rightarrow R = Q^T A = \begin{bmatrix} \underline{q}_1^T \\ \underline{q}_2^T \\ \underline{q}_3^T \end{bmatrix} \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \underline{a}_3 \end{bmatrix}$.

$$R = \begin{bmatrix} \underline{q}_1^T \underline{a}_1 & \underline{q}_1^T \underline{a}_2 & \underline{q}_1^T \underline{a}_3 \\ \underline{q}_3^T \underline{a}_1 & \underline{q}_2^T \underline{a}_2 & \underline{q}_2^T \underline{a}_3 \\ \underline{q}_3^T \underline{a}_1 & \underline{q}_3^T \underline{a}_2 & \underline{q}_3^T \underline{a}_3 \end{bmatrix} = \begin{bmatrix} \underline{q}_1^T \underline{a}_1 & \underline{q}_1^T \underline{a}_2 & \underline{q}_1^T \underline{a}_3 \\ 0 & \underline{q}_2^T \underline{a}_2 & \underline{q}_2^T \underline{a}_3 \\ 0 & 0 & \underline{q}_3^T \underline{a}_3 \end{bmatrix}.$$

$$\begin{aligned} A &= QR \\ &= \begin{bmatrix} \underline{q}_1 & \underline{q}_2 & \underline{q}_3 \end{bmatrix} \begin{bmatrix} \underline{q}_1^T \underline{a}_1 & \underline{q}_1^T \underline{a}_2 & \underline{q}_1^T \underline{a}_3 \\ 0 & \underline{q}_2^T \underline{a}_2 & \underline{q}_2^T \underline{a}_3 \\ 0 & 0 & \underline{q}_3^T \underline{a}_3 \end{bmatrix}. \end{aligned}$$

The least squares setup becomes:

$$\begin{aligned} & A^T A \underline{\hat{x}} = A^T \underline{b} \Rightarrow R^T Q^T Q R \underline{\hat{x}} = R^T Q^T \underline{b} \Rightarrow R^T R \underline{\hat{x}} = R^T Q^T \underline{b} \\ & \Rightarrow R \underline{\hat{x}} = Q^T \underline{b} \end{aligned}.$$

# V. DETERMINANTS

When $A^{-1}$ exists, $\det\left(A^{-1}\right)=1/\det(A)$.

Determinant=product of pivots.

1. The determinant of the $n\times n$ identity matrix is 1.
2. The determinant changes sign when two rows (or two columns) are exchanged. For permutation matrices, $|P|=1$ for an even number of exchanges and $|P|=-1$ for an odd number of exchanges.

$\det(A+B)\neq\det(A)+\det(B)$.

$\det(A)=|A|=\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}$. $\det(AB)=\det(A)\det(B)$. $\det\left(A^T\right)=\det(A)$.

1. If a row is multiplied by a number, while other rows are not changed, the determinant is multiplied by the same number:

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix}=t\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

If a row of the first matrix is added to the corresponding row of another matrix, while other rows are not changed, determinants add:

$$\begin{vmatrix} a & b \\ c+g & d+h \end{vmatrix}=\begin{vmatrix} a & b \\ c & d \end{vmatrix}+\begin{vmatrix} a & b \\ g & h \end{vmatrix}.$$

$\det(tA)=t^n\det(A)$.

2. If two rows are equal, the determinant is zero.
3. Subtracting a multiple of one row from another row leaves the determinant unchanged. $\det(A)=\det(U)$.
4. A matrix with a row of zeros has zero determinant.
5. Determinant of a triangular matrix equals the product of diagonal elements.
6. Matrix with zero det is singular

First $k$ pivots come from a $k\times k$ matrix $A_k$ in the top left corner of $A$.

$$k^{\text{th}}\text{ pivot}=\frac{|A_k|}{|A_{k-1}|}.$$

$\det A =$ sum over all $\mathbf{n}!$ column permutations $P = (\alpha, \beta, \ldots, \omega)$

$$= \sum (\det P) a_{1\alpha} a_{2\beta} \cdots a_{n\omega} = \textbf{BIG FORMULA}.$$

$$\det A = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

## V.1. Determinant by Cofactors

The cofactor of $a_{11}$ is $C_{11} = a_{22}a_{33} - a_{23}a_{32}$. You can see it in this splitting:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}.$$

We are still choosing ***one entry from each row and column***. Since $a_{11}$ uses up row 1 and column 1, that leaves a 2 by 2 determinant as its cofactor.

As always, we have to watch signs. The 2 by 2 determinant that goes with $a_{12}$ looks like $a_{21}a_{33} - a_{23}a_{31}$. But in the cofactor $C_{12}$, *its sign is reversed*. Then $a_{12}C_{12}$ is the correct 3 by 3 determinant. The sign pattern for cofactors along the first row is plus-minus-plus-minus. ***You cross out row*** 1 ***and column*** $j$ ***to get a submatrix*** $M_{1j}$ ***of size*** $n-1$. Multiply its determinant by $(-1)^{1+j}$ to get the cofactor:

The cofactors along row 1 are $C_{1j} = (-1)^{1+j} \det M_{1j}$.

**The cofactor expansion is** $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$. (11)

## V.2. Cramer's Rule

$$A\underline{x} = \underline{b} \Rightarrow A[\underline{x} \quad \underline{e}_2 \quad \underline{e}_3] = [\underline{b} \quad \underline{a}_2 \quad \underline{a}_3] = B_1.$$

$$A\begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix} = B_1.$$

$$\det(A)x_1 = \det(B_1) \Rightarrow x_1 = \frac{\det(B_1)}{\det(A)}.$$

To get $x_2$, put $\underline{x}$ in the second column of $I \Rightarrow x_2 = \dfrac{\det(B_2)}{\det(A)}$.

Similarly, $x_j = \dfrac{\det(B_j)}{\det(A)}$.

Let $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Rightarrow x_1 = \dfrac{\begin{vmatrix} 6 & 3 \\ 2 & 2 \end{vmatrix}}{5} = \dfrac{6}{5}, x_1 = \dfrac{\begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix}}{5} = \dfrac{2}{5}$.

## V.2.A. Finding Inverse

Let $A^{-1} = G$

$$A\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A\begin{bmatrix} g_{11} \\ g_{21} \\ g_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow A\begin{bmatrix} g_{11} & 0 & 0 \\ g_{21} & 1 & 0 \\ g_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{33} & a_{33} \end{bmatrix}.$$

$$\det(A) g_{11} = \begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{33} & a_{33} \end{vmatrix} \Rightarrow g_{11} = \frac{\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{33} & a_{33} \end{vmatrix}}{\det(A)} = \frac{C_{11}}{\det(A)}.$$

$$g_{ij} = \frac{C_{ji}}{\det(A)} \Rightarrow A^{-1} = \frac{C^T}{\det(A)}.$$

The triangle with corners $(x_1, y_1)$ and $(x_2, y_2)$ and $(x_3, y_3)$ has area $= \dfrac{\text{determinant}}{2}$:

**Area of triangle** $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ Area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$ when $(x_3, y_3) = (0, 0)$.

**DEFINITION** The ***cross product*** of $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ is a vector

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\boldsymbol{i} + (u_3v_1 - u_1v_3)\boldsymbol{j} + (u_1v_2 - u_2v_1)\boldsymbol{k}. \quad (10)$$

*This vector is perpendicular to* $\boldsymbol{u}$ *and* $\boldsymbol{v}$. The cross product $\boldsymbol{v} \times \boldsymbol{u}$ is $-(\boldsymbol{u} \times \boldsymbol{v})$.

Let $\underline{u} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}, \underline{v} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \Rightarrow \underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = -1\underline{i} - 5\underline{j} - 7 \Rightarrow \underline{u} \times \underline{v} = \begin{bmatrix} -1 \\ -5 \\ -7 \end{bmatrix}$.

$\underline{u} \times \underline{v} \perp \underline{u}, \underline{v}$ . $\underline{u} \times \underline{v} = -\underline{v} \times \underline{u}$ . $\underline{u} \times \underline{u} = 0$ .

$\|\underline{u} \times \underline{v}\| = \|\underline{u}\|\|\underline{v}\|\sin(\theta) =$ area of the parallelogram with sides $\underline{u}$ and $\underline{v}$..

## V.2.B. Volume

$(\underline{u} \times \underline{v}) \cdot \underline{w} = \begin{vmatrix} \underline{w}^T \\ \underline{u}^T \\ \underline{v}^T \end{vmatrix} = \begin{vmatrix} \underline{u}^T \\ \underline{v}^T \\ \underline{w}^T \end{vmatrix} =$ volume of the $\underline{u}, \underline{v}, \underline{w}$ box.

# VI. EIGENVALUES AND EIGENVECTORS

Generally, vectors change direction when left-multiplied by a matrix.

$A\underline{x} = \lambda \underline{x}$ means that the vector does not change direction when multiplied by the matrix.

$\lambda$ is a eigenvalue of the matrix. $\underline{x}$ is an eigenvector.

$$A\underline{x} - \lambda \underline{x} = 0$$
$$A\underline{x} - \lambda I\underline{x} = 0$$
$$\det(A - \lambda I) = 0$$

$(A - \lambda I)\underline{x} = 0 \Rightarrow \underline{x} \in \mathcal{N}(A - \lambda I)$.

Let $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \Rightarrow \begin{vmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{vmatrix} = (0.8-\lambda)(0.7-\lambda) - 0.06 = 0 \Rightarrow 0.5 - 0.15\lambda + \lambda^2 = 0$.

$0.5 - 0.15\lambda + \lambda^2 = 0 \Rightarrow (1-\lambda)(0.5-\lambda) = 0 \Rightarrow \lambda_{1,2} = 1, 0.5$.

To find the eigenvectors, note that $A - \lambda_1 I$ and $A - \lambda_2 I$ are singular.

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0 \Rightarrow -2x_{11} + 3x_{12} = 0 \Rightarrow x_{11} = \frac{3}{2}x_{12} \Rightarrow \underline{x}_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}.$$

$$\begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0 \Rightarrow 3x_{21} + 3x_{22} = 0 \Rightarrow x_{21} = -x_{22} \Rightarrow \underline{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Eigenvectors are not unique.

$$A\underline{x}_i = \lambda_i \underline{x}_i \Rightarrow A(A\underline{x}_i) = \lambda_i A\underline{x}_i = \lambda_i^2 \underline{x}_i \Rightarrow A^2\underline{x}_i = \lambda_i^2 \underline{x}_i \Rightarrow A^n \underline{x}_i = \lambda_i^n \underline{x}_i \Rightarrow$$

$A^n$ has same eigenvectors as $A$. $A^n$ has $\left\{\lambda_i^n\right\}$ eigenvalues.

$A\underline{x}_i = \lambda_i \underline{x}_i \Rightarrow \lambda_i \underline{x}_i$ is a linear combination of the columns of $A$. Each column of $A$ is a linear combination of the eigenvectors.

In the above, $\underline{a}_1 = \underline{x}_1 + 0.2\underline{x}_2$, $\underline{a}_2 = \underline{x}_1 - 0.3\underline{x}_2$.

Note that $A\underline{a}_1 = A\left(\underline{x}_1 + 0.2\underline{x}_2\right) = \lambda_1 \underline{x}_1 + 0.2\lambda_2 \underline{x}_2 = \underline{x}_1 + 0.1\underline{x}_2 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} \Rightarrow$

$A^{99}\underline{a}_1 = \underline{x}_1 + 0.2\lambda_2^{99}\underline{x}_2 = \text{column 1 of } A^{100} \approx \underline{x}_1$.

$\underline{x}_1$ is a steady state, while $\underline{x}_2$ is a decaying mode.

All entries of $A$ are positive, and the sum of every column is 1. This is a Markov matrix.

When columns add to 1, $\lambda = 1$ is an eigenvalue.

When the matrix is singular, $\lambda = 0$ is an eigenvalue.

When the matrix is symmetric, eigenvectors are orthogonal.

$A = LU$. Eigenvalues of $U$ are the pivots; and they are not the eigenvalues of $A$.

The product of the eigenvalues equals the determinant.

The sum of the eigenvalues equals the sum of the diagonal entries (trace).

Example: Rotation matrix $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_{1,2} = \pm i \Rightarrow \underline{x}_{1,2} = \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} i \\ 1 \end{bmatrix}$.

$Q$ is an orthogonal matrix $\Rightarrow |\lambda_i| = 1$.

$Q$ is a skew symmetric matrix $\Rightarrow \lambda_i$ is pure imaginary.

## VI.1. Diagonalizing a Matrix

Let the $n \times n$ matrix $A$ have $n$ linearly independent eigenvectors. Let $S$ have its columns as the eigenvectors of $A$.

$S = [\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_n]$. $AS = A[\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_n] = [\lambda_1 \underline{x}_1 \quad \lambda_2 \underline{x}_2 \quad \cdots \quad \lambda_3 \underline{x}_n]$.

$$[\lambda_1 \underline{x}_1 \quad \lambda_2 \underline{x}_2 \quad \cdots \quad \lambda_3 \underline{x}_n] = [\underline{x}_1 \quad \underline{x}_2 \quad \cdots \quad \underline{x}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = S\Lambda .$$

$AS = S\Lambda \Rightarrow A = S\Lambda S^{-1} \Rightarrow A^n = S\Lambda^n S^{-1}$.

Let $A = \begin{bmatrix} 1 & 5 \\ 0 & 6 \end{bmatrix} \Rightarrow \lambda_{1,2} = 1, 6 \Rightarrow \underline{x}_{1,2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. These are independent.

$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, S^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \Lambda = \begin{bmatrix} 1 & \\ & 6 \end{bmatrix}$.

$$A=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & \\ & 6\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\Rightarrow A^k=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1^k & \\ & 6^k\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 6^k-1\\ 0 & 6^k\end{bmatrix}.$$

Any matrix that has no repeated eigenvalues has independent eigenvectors, and it can be diagonalized.

Let $\underline{u}_{k+1}=A\underline{u}_k \Rightarrow \underline{u}_k=A^k\underline{u}_0=S\Lambda^k S^{-1}\underline{u}_0$.

Let $\underline{c}=\begin{bmatrix}c_1\\ c_2\\ \vdots\\ c_n\end{bmatrix}\Rightarrow \underline{u}_0=c_1\underline{x}_1+c_2\underline{x}_2+\cdots+c_n\underline{x}_n=S\underline{c}\Rightarrow \underline{c}=S^{-1}\underline{u}_0$.

$$\begin{aligned}\underline{u}_k &= A^k\underline{u}_0=S\Lambda^k S^{-1}\underline{u}_0\\ &= c_1\lambda_1^k\underline{x}_1+c_2\lambda_2^k\underline{x}_2+\cdots+c_n\lambda_n^k\underline{x}_n\end{aligned}$$

If $AB=BA$, then $A$ and $B$ share the same $S$.

## VI.2. Applications to Differential Equations

Note that $\frac{d}{dt}\left(e^{\lambda t}\right)=\lambda e^{\lambda t}$.

$\frac{du(t)}{dt}=\lambda u(t)\Rightarrow u(t)=Ce^{\lambda t}$. Note that $u(0)=C\Rightarrow u(t)=u(0)e^{\lambda t}$.

Consider $\frac{d}{dt}\underline{u}(t)=A\underline{u}(t)$.

Let $\underline{u}(t)=e^{\lambda t}\underline{x}$, where $\lambda$ is an eigenvalue of $A$ and $\underline{x}$ is an eigenvector of $A$: $A\underline{x}=\lambda\underline{x}$.

$\frac{d}{dt}\left(u(t)\right)=\lambda e^{\lambda t}\underline{x}=Ae^{\lambda t}\underline{x}=A\underline{u}(t)$.

Solve $\frac{d}{dt}\underline{u}(t)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\underline{u}(t), \underline{u}(0)=\begin{bmatrix}4\\ 2\end{bmatrix}$.

$\lambda_{1,2}=1,-1$, $\underline{x}_{1,2}=\begin{bmatrix}1\\ 1\end{bmatrix},\begin{bmatrix}1\\ -1\end{bmatrix}\Rightarrow \underline{u}_1(t)=e^t\begin{bmatrix}1\\ 1\end{bmatrix},\underline{u}_2(t)=e^{-t}\begin{bmatrix}1\\ -1\end{bmatrix}$.

$$\underline{u}(t) = C\underline{u}_1(t) + D\underline{u}_2(t) = \begin{bmatrix} Ce^{t} + De^{-t} \\ Ce^{t} - De^{-t} \end{bmatrix} \Rightarrow \begin{bmatrix} C+D \\ C-D \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \Rightarrow C = 3, D = 1.$$

## VI.3. Second Order Equations

$my'' + by' + k = 0$. Let $y(t) = e^{\lambda t} \Rightarrow m\lambda^2 + b\lambda + k = 0$.

If $\lambda_1 \neq \lambda_2 \Rightarrow y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$.

Let $m = 1 \Rightarrow \begin{matrix} \frac{dy}{dt} = y' \\ \frac{dy'}{dt} = -ky - by' \end{matrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \Rightarrow \frac{d\underline{u}}{dt} = A\underline{u}$.

$$\begin{vmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + b\lambda + k = 0.$$

$\underline{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$. $\underline{u}(t) = c_1 e^{\lambda_1 t}\underline{x}_1 + c_2 e^{\lambda_2 t}\underline{x}_2$.

Consider the equation: $\begin{matrix} y'' + y = 0 \\ y(0) = 1,\ y'(0) = 0 \end{matrix} \Rightarrow \begin{matrix} y'' = -y \\ \underline{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{matrix}$.

$$\begin{matrix} \frac{dy}{dt} = y' \\ \frac{dy'}{dt} = -y \end{matrix} \Rightarrow \frac{d}{dt}\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y \\ y' \end{bmatrix} \Rightarrow \lambda_1 = i, \lambda_2 = -i \Rightarrow \underline{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \underline{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

$$\underline{u}(t) = c_1 e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix}.\ \underline{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \underline{u}(t) = \frac{1}{2}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

**Stability** $A$ is ***stable*** and $\boldsymbol{u}(t) \to \mathbf{0}$ when all eigenvalues have ***negative real parts.*** The 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must pass two tests:

| | |
|---|---|
| $\lambda_1 + \lambda_2 < 0$ | The trace $T = a + d$ must be negative. |
| $\lambda_1 \lambda_2 > 0$ | The determinant $D = ad - bc$ must be positive. |

If the $\lambda$'s are complex numbers, they must have the form $r + is$ and $r - is$. Otherwise $T$ and $D$ will not be real. The determinant $D$ is automatically positive, since $(r+is)(r-is) = r^2 + s^2$. The trace $T$ is $r + is + r - is = 2r$. So a negative trace means that the real part $r$ is negative and the matrix is stable. Q.E.D.

Eigenvalues of $f(A)$ are $f(\lambda)$. Eigenvalues of $e^{\Lambda t}$ are $\left\{e^{\lambda_i t}\right\} \Rightarrow e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$.

$e^{At} = Se^{\Lambda t}S^{-1}$.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^t & 0.5e^{3t} - 0.5e^t \\ 0 & e^{3t} \end{bmatrix}$.

## VI.4. Symmetric Matrices:

Eigenvalues of $A^T$ and $A$ are the same.

Eigenvectors of a real symmetric matrix (when the eigenvalues are different) are always perpendicular.

For a symmetric matrix with real number entries, the eigenvalues are real numbers and it's possible to choose a complete set of eigenvectors that are perpendicular (or even orthonormal).

Spectral Theorem: If $A = A^T \Rightarrow S = Q \Rightarrow A = Q\Lambda Q^{-1} = Q\Lambda Q^T$.

The eigenvalues of $A + bI$ are just $b$ more than the eigenvalues of $A$. This is true for all matrices.

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow \underline{x}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix}, \underline{x}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}.$$

$A = Q\Lambda Q^T = \begin{bmatrix} \underline{x}_1 & \underline{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \end{bmatrix} = \lambda_1 \underline{x}_1 \underline{x}_1^T + \lambda_2 \underline{x}_2 \underline{x}_2^T$. Generalize - $\underline{x}_i \underline{x}_i^T$: projection matrix.

*product of pivots = determinant = product of eigenvalues.*

*The number of positive eigenvalues of a symmetric matrix equals the number of positive pivots.*

*Symmetric matrices are always diagonalizable.*

***Positive definite matrices***

A *positive definite matrix* is a symmetric matrix $A$ for which all eigenvalues are positive. A good way to tell if a matrix is positive definite is to check that all its pivots are positive.

Example: 2x2 symmetric matrix with positive pivots.

$A\underline{x} = \lambda \underline{x} \Rightarrow \underline{x}^T A\underline{x} = \lambda \underline{x}^T \underline{x} =$ positive scalar.

When $A$ is positive definite, eigenvectors are independent and $\underline{x}^T A\underline{x} > 0$ for all $\underline{x}$.

If $A$ and $B$ are PD then so is $A + B$.

If $R$ (possibly rectangular) has independent columns, then $A = R^T R$ is PD because:

$\underline{x}^T A\underline{x} = \underline{x}^T R^T R\underline{x} = (R\underline{x})^T R\underline{x} > 0$.

***When a symmetric matrix has one of these five properties, it has them all:***

1. All ***n pivots*** are positive.
2. All ***n upper left determinants*** are positive.
3. All ***n eigenvalues*** are positive.
4. $x^T Ax$ is positive except at $x = \mathbf{0}$. This is the ***energy-based*** definition.
5. $A$ equals $R^T R$ for a matrix $R$ with ***independent columns***.

## VI.5. Positive Semidefinite Matrices

$\underline{x}^T A\underline{x} = 0$ (for eigenvectors only)

## VI.6. Similar Matrices

Let $B = M^{-1}AM \Rightarrow AM = MB \Rightarrow AM\underline{x}_B = MB\underline{x}_B = \lambda_B M\underline{x}_B \Rightarrow \lambda_B$ is an eigenvalue of $A$. We say that $A$ and $B$ are similar. Eigenvectors of $A$ are $M\underline{x}_B$.

***Singular Value Decomposition (SVD)***

Let the eigenvectors of $A^T A$ be $\{\underline{v}_i\}$, and the eigenvectors of $AA^T$ be $\{\underline{u}_i\}$. Both sets of eigenvectors can be chosen to be orthonormal.

$U = [\underline{u}_1 \quad \underline{u}_2 \quad \cdots \quad \underline{u}_r]$, $V = [\underline{v}_1 \quad \underline{v}_2 \quad \cdots \quad \underline{v}_r]$.

Let $A=\begin{bmatrix}2 & 2\\ -1 & 1\end{bmatrix} \Rightarrow A^T A=\begin{bmatrix}5 & 3\\ 3 & 5\end{bmatrix} \Rightarrow \underline{v}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix}, \underline{v}_2=\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\ 1\end{bmatrix},\ \sigma_1^2=\left(2\sqrt{2}\right)^2=8$.

$A\underline{v}_1=\begin{bmatrix}2\sqrt{2}\\ 0\end{bmatrix} \Rightarrow \underline{u}_1=\begin{bmatrix}1\\ 0\end{bmatrix}, A\underline{v}_2=\begin{bmatrix}0\\ \sqrt{2}\end{bmatrix} \Rightarrow \underline{u}_2=\begin{bmatrix}0\\ 1\end{bmatrix},\ \sigma_2^2=\left(\sqrt{2}\right)^2=2$.

$$A=U\Sigma V^T=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}.$$

# VII. COMPLEX MATRICES

$A^H = (A^*)^T$.

$\|\underline{x}\|^2 = \underline{x}^H \underline{x} = \sum_{i=1}^{n} x_i^* x_i = \sum_{i=1}^{n} |x_i|^2$. This is the inner product.

If $A^H = A$, then $A$ is Hermitian and $\underline{z}^H A \underline{z}$ is real.

***Every eigenvalue of a Hermitian matrix is real.***

***The eigenvectors of a Hermitian matrix are orthogonal.***

***A unitary matrix U is a (complex) square matrix that has orthonormal columns.*** $U^H U = I$

$U = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$ is unitary.